PLANE GEOMETRY

HAWKES-LUBY-TOUTON

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PLANE GEOMETRY

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PREFACE

Although the study of geometry is important from an informational point of view, it is generally recognized that a genuine mastery of the subject means real achievement in the solution of original exercises. The chief aim of the authors in preparing this text has been to give such assistance to students as will stimulate insight and develop power to solve exercises of gradually increasing difficulty.

The content and organization of the first book of geometry are determining factors in the student's progress in the subject. The first ten or twenty theorems determine whether a student will grasp quickly the general trend of the subject or remain bewildered for an indefinite period. The simplicity and directness of the early theorems and the order in which they are presented are elements of the highest importance in an effective introduction. These elements were given the most careful consideration in the preparation of this text. If the first theorem is used in proving the second and the second in proving the third, and so on, the student will soon see a reason for mastering the content of each theorem. But if the initial theorem is used for the first time in the fifteenth theorem, and the second is used next in the eighth, the first month's work will give the impression that geometry deals with unrelated facts which lead nowhere. Such an arrangement of the theorems of geometry will hamper even the strongest students, and will make progress almost impossible for the less capable.

In this text no time is wasted on side issues or on discussive explanations, but the student is brought by a direct route to the theorems on angle sums of polygons, the theorems on parallel lines, and those on parallelograms. Before reaching Theorem 13 he meets theorems on angle sums, the one topic in Book I which has varied numerical applications. Exercises based on angle sums are presented on page 35, at which point the student begins his really independent work upon numerical exercises. Having acquired some ability in solving numerical exercises he is prepared to begin the devising of general proofs involving congruent triangles.

The most important single method of elementary geometry is the use of congruent triangles. The best experience of teachers has shown that practice with this method should begin early and last long. Superposition is used here only when unavoidable, and the more difficult topics, such as inequalities and indirect methods, are deferred until near the end of the book.

Attention is invited to certain general features of the text. For example, axioms, postulates, and definitions are given when needed, not before. The usual group of theorems on proportion are distributed, each one being presented at the proper point. An important feature is the placing of a few exercises involving suitable numerical and general applications of a theorem immediately after it. Many exercises are presented which are designed to correlate geometry with algebra, and to compel the student to use his knowledge of fractions, equations, and radicals. A large number of pertinent and stimulating queries are given which demand only direct answers. Occasionally queries are used to develop a specific topic such as loci, but usually they are designed to broaden and perfect the student's knowledge of details

supplementary to the theorems and exercises. The parallel-column arrangement of the demonstrations has been chosen in the belief that it is an aid to clear thinking, and that it places unusual and proper emphasis on the necessity of stating a reason for each assertion. Frequently, to encourage independent thinking, reasons have been omitted and " Why ? " has been inserted. This is done sparingly at first, but later on with an increasing frequency graduated to the student's growing knowledge of the subject.

Certain special features have received careful treatment. Emphasis on right triangles having one angle 30°, 45°, or 60°, which begins with Theorem 24 of Book I, is continued over into other books, especially Book III. This is justified by the experience of every teacher of trigonometry and by the specific recommendation of the Society for the Promotion of Engineering Education. In fact, no theorem has been omitted which any important educational body considers necessary. In Book II particular attention is called to the emphasis placed on constructions and loci and their interrelations. An appreciation of these relations makes it clear why real constructional work should not come earlier in a course on demonstrative geometry. If it does, the results must be assumed without proof. It seems far more desirable to assume, in the few cases needed, the existence of certain lines and figures, and to defer the actual construction of them until it is possible to prove that the method used is correct. Full advantage has been taken of the unusual opportunity offered by Book V for useful and necessary constructions. The presentation of the matter of that book is one of the unique features of the text. The graphical methods of solution given on pages 218 and 219 satisfy the most rigorous criteria for practical problems. A further illustration of the

importance of graphical methods is the work of page 281 on areas. Attention is invited to Theorem 2, Book III, a simple proof of which is given not involving the use of the theorem of limits. In the other theorems usually proved by the method of limits the meaning of the theorem has been brought out by illustration and the student convinced of its truth without any attempt at proof. This is in accord with a widespread feeling that the method of limits is unprofitable for the majority of American students when they first study the plane geometry. The plan also avoids the common error of designating as a proof a line of thought which is not a proof.

One hundred and eighty additional exercises, grouped according to the books on which they depend, will be found at the end of the text.

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PLANE GEOMETRY

THE ORIGIN OF GEOMETRY

The elementary geometry of to-day is mainly due to the genius of the Greeks. The foundation for their work, however, was obtained from the Babylonians and the Egyptians.

The Babylonian knowledge of geometry was developed, in part at least, through the necessity of constructing figures having religious significance. These figures involved triangles, quadrilaterals, circles, and inscribed regular hexagons. Their value 3 for π was much less accurate than 3.1604, which was used by the Egyptians probably as early as 3000 B.C.

The Egyptians were great builders. We read that Menes, their first king, who built the temple of Ptah at Memphis, also constructed a great reservoir and even changed the course of the Nile. The ruins of many temples, the Sphinx, and the great pyramids of Gizeh all attest considerable insight into geometric relations. The geometry they developed was applied to the calculation of the contents of granaries, to the laying out of right angles, to obtaining north-and-south lines for their temples and pyramids, and to carrying out land surveys. Herodotus, the Greek historian, who traveled in Egypt, says that the overflow of the Nile necessitated an annual resurvey of the land along its banks for the double purpose of determining ownership and of justly levying taxes. The word *geometry* itself is of Greek origin and signifies "earth measurement."

Our information concerning the Egyptians goes back with certainty to 1700 B.C. and with great probability to 3000 B.C. We know that two thousand, perhaps three thousand, years before our era they had formulas — a few of which were incorrect — for the areas of triangles, rectangles, parallelograms, and trapezoids, and fairly accurate knowledge of the area of a circle. They knew also that a triangle is a right triangle if its sides are 3, 4, and 5 units respectively. Such geometry as they had grew out of their practical needs and was embodied in working rules. It was very useful, but vastly inferior to the scientific geometry afterwards developed by the Greeks. In judging the intellectual attainments of any people, however, we must remember that first advances in any science are the most difficult. Other peoples had the same practical needs as the Egyptians but developed no geometry to meet them. The Aztecs of Mexico, for example, were skillful artificers in silver and gold, and they transported huge blocks of stone from distant quarries to build their splendid temples, but their imagination never led them to construct a vehicle on wheels, — one of the simplest applications of a circle, but undoubtedly the most useful, which has ever been made.

BOOK I

1. Introduction. Geometry is a science which treats of the properties and measurements of flat figures, such as angles, triangles, and circles, and of solid figures, such as cubes, cones, and pyramids. The subject is not made up of entirely new matter, for every student has had some experience with geometric facts in his previous study. It will be found that geometry uses arithmetic and algebra freely, though it is not primarily concerned either with numbers or with the equation.

2. Fundamental ideas. Before taking up the part of geometry which is entirely new, we must make more precise our knowledge of some facts and relations already partially known. Certain familiar terms must be carefully defined, certain ideas illustrated, and certain fundamental truths examined and accepted as a basis for further work.

3. Definition. A *definition* is a statement explaining the meaning of a word or phrase.

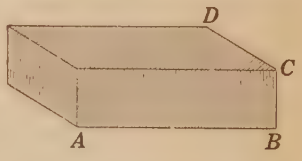
Every definition should be clear, complete, and in terms more simple and familiar than the one defined. The essential qualities of the term defined should be made so clear that whatever has these qualities, or does not have them, can easily be recognized.

Not every term can be defined. The word *space*, for example, denotes a concept extremely hard to put in words. Terms which are difficult to define accurately can usually be made clear by description or by illustration or by both. This method is necessary with the term *straight line*. Then there will, of course, be various terms, such as *length*, *breadth*, and

thickness, which need not be defined, as the ideas they convey are already understood with sufficient clearness.

4. Solids. A board, a piece of chalk, or an iron bar is an example of a physical solid. If a board be moved from one place to another, the space it has occupied at any instant is an example of a geometric solid. A physical solid can be seen and touched; a geometric solid cannot — it is purely a mental concept. The science of physics is a study of the nature and properties of the material of which physical solids are constituted, while geometry is a study of the size and shape of solids but not of the matter composing them.

5. Dimensions. The smooth block $ABCD$ extends in three principal directions; that is, it has three *dimensions*, — length AB, breadth (or width) CD, and thickness (or height, depth, or altitude) BC. In like manner every solid has three dimensions.



6. Surfaces. Each of the flat faces bounding $ABCD$ is called a *surface*. The boundaries of a geometric solid are also called surfaces. A better illustration of a geometric surface is one's shadow on a smooth sidewalk.

A surface has two dimensions, length and breadth. Even the thinnest piece of paper is not a surface; it is a solid, having length and breadth and a small but measurable thickness.

7. Lines. Any two surfaces of $ABCD$ meet in an edge. In the geometric solid the edges are called *lines*. We may think of the intersection of any two surfaces as a line.

A geometric line has one dimension only, length. Of course any line we may draw or engrave has breadth. Since over 20,000 parallel lines have been cut by a diamond point on a steel plate one inch long, the breadth of such lines is very minute, yet they give only an approximate idea of what is meant by a geometric line.

8. Points. Any three edges of $ABCD$ which meet form what is called a *corner*. In the geometric solid this is called a *point*. We may think of the intersection of any *two* lines as a point.

Any small dot which we can make on paper, such as the period at the end of this sentence, has a measurable size. Such a point, however small, is only a crude representation of a geometric point.

9. Magnitudes. A *magnitude* is anything which is measurable or which may be thought of as measurable.

Anything of which we may ask, How much? is a magnitude. Lines, surfaces, and solids are examples of magnitudes.

10. The geometric magnitudes generated by motion. If a point moves, it generates (describes or traces) a line; if a line moves (except along itself), it generates a surface; if a surface, such as a triangle or a square, moves (except along itself), it generates a solid.

11. Straight line. If a weight is suspended by a fine piano wire, the position taken by the wire is a good physical representation of a *straight line.*

When the word *line* is used alone, usually a straight line is meant.

A straight line is considered as extending indefinitely in both directions.

A line beginning at a definite point and extending indefinitely in one direction only is a *ray.*

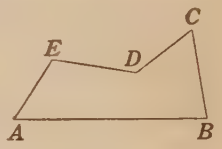
A part of a line lying between any two of its points is called a *line-segment.* Thus AB is a segment of KR.

12. Plane. A *plane* is a surface such that a straight line joining any two of its points lies wholly within the surface.

In finishing up concrete walks a workman often lays a straightedge across the surface in several directions to test its flatness; that is, to determine if it is a plane.

13. Geometric figures. A point, a line, a surface, a solid, or any combination of them is called a *geometric figure.*

A *plane figure* is any geometric figure which lies in a plane (flat) surface.



A triangle and a square are examples of plane figures.

Plane geometry is concerned with the study of plane figures only.

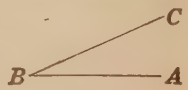
A *rectilinear figure* is a figure bounded by straight lines. *ABCDE* is a rectilinear figure.

14. Boundaries. The *boundaries* of a geometric figure are the points, lines, curves, or surfaces which limit it.

Thus the boundaries of a line-segment are its two end points, the boundaries of a surface are lines, and the boundaries of a solid are surfaces.

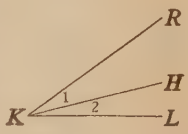
15. Angle. A plane *angle* (symbol $\angle$) is the figure formed by two rays which meet.

The two rays are called the *sides* of the angle and the point at which they meet is called the *vertex.*



Thus AB and CB are the sides of the angle ABC and B is the vertex.

In naming an angle, the vertex letter is named between the other two, thus: the angle ABC or the angle CBA. It is also proper to name an angle by the vertex letter alone if no other angle has the same vertex. Thus we may call angle CBA angle B. In the adjacent figure, to speak of the angle K would be indefinite, as any one of three angles has the vertex K. Whenever two or more angles have the same vertex we must name each angle by three letters, or we can write a small letter or number within the angle near its vertex and name each angle by one character. Thus angle RKH may be called angle 1 and angle HKL may be called angle 2.



16. Triangle. A *triangle* (symbol $\triangle$) is a portion of a plane bounded by three straight lines.

Every triangle is regarded as having *six parts*: three sides and three angles.

17. Axiom. An *axiom* is a general statement which is accepted as true without proof.

Axioms are truths so simple that we either cannot prove them or do not care to do so. An example of a mathematical axiom is: *If the same number be added to each member of an equation, the result is an equation.*

18. Postulate. A *postulate* is a geometric statement which is accepted as true without proof.

19. *Postulate I.* *There is only one straight line through two points.*

20. *Postulate II.* *Any geometric figure may be moved from one place to another without changing its size or shape.*

The postulates I and II are needed at once in Theorem I. Other postulates and axioms will be stated as the need for them arises.

21. Theorem. A *theorem* is a statement of a truth which is to be proved.

An example of a theorem is: The square of the sum of two numbers is the square of the first plus twice their product plus the square of the second.

22. Proof. A *proof*, or a *demonstration*, is an argument or an orderly arrangement of statements with reasons by which we show the truth of an assertion.

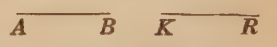
In the course of a geometrical proof we proceed from inference to inference, and we may appeal to any or all of the following: to the definition of a term previously agreed on; to an axiom or a postulate already stated; to a theorem already proved.

Sometimes we appeal directly or indirectly to common sense, — to obvious facts, or to ideas or definitions which are so universally accepted that a detailed statement of them is not required.

23. Equality of two geometric magnitudes. If two magnitudes of the same kind, like two angles or two distances, are measured and their measures are expressed in terms of a common unit by *the same number*, the two magnitudes are equal.

In geometry, however, it is often desirable to prove two magnitudes of the same kind (two angles, or two triangles, or two other plane figures) equal *without measuring them*. One method of doing this consists in proving that the boundaries of the two magnitudes may be made to coincide.

Thus the line-segment AB is equal to the line-segment KR provided that when A is placed on K, B at the same time can be placed on R.



Similarly, the angle ABC is equal to the angle KRL if B can be placed on R and BA upon RK, and if at the same time BC can be placed upon RL.

Note that the size of an angle depends on the size of the opening between its sides (see § 15) and not on their length. In showing that angle B equals angle R, BA, which is represented as shorter than RK, falls upon RK but does not *coincide* with it.

Lastly, the triangle ABC is equal to the triangle KRL if when A is placed on K, B at the same time may be placed on R and C on L. For then AB would coincide with KR, BC with RL, and CA with LK; that is, the boundaries of the triangle ABC would coincide with the boundaries of the triangle KRL. (See Postulate I.)

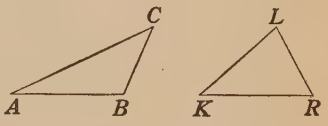
24. Congruence. Two geometric magnitudes are *congruent* if their boundaries can be made to coincide.

Congruent figures, therefore, are always equal, but equal figures are not necessarily congruent.

Magnitudes having *only one dimension*, such as line-segments, are *congruent* if they are *equal*. With all such magnitudes *equal* and *congruent* mean the same thing. Magnitudes having two or more dimensions may be equal without being congruent.

Thus the area of the triangle ABC (that is, the number of square units in the surface) is equal to the area of the triangle KRL, but the two triangles are not congruent.

A triangle and a square, each containing forty square inches, are equal, but they are not congruent.

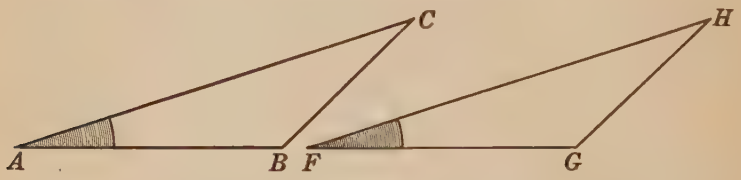


If two figures are equal they have the same size but not necessarily the same shape, while if two figures are congruent they have the same size and also the same shape.

We shall now take up the first theorem and by reasoning about it prove its truth. Other theorems depending on the first will follow, and by their aid we shall prove still others. Truth after truth will thus be demonstrated, all forming a closely connected whole. Some of these truths without their proof the student has met and used before. It may be of assistance to mention a few of the theorems which are proved in geometry. We shall prove that the sum of the angles of any triangle is 180°; that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides; that the area of any circle is equal to the square of the radius multiplied by 3.1416; that the volume of any pyramid is one third the area of the base times the altitude; and that the volume of any sphere is four thirds of the product of the cube of the radius and 3.1416.

Theorem 1

25. *If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the two triangles are congruent.*



Given the triangles *ABC* and *FGH*, in which *AB* equals *FG*, *AC* equals *FH*, and the angle *A* equals the angle *F*.

To prove *that* $\triangle ABC$ *is congruent to* $\triangle FGH$.

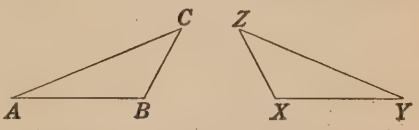
Proof

Statement	Reason
1. Place $\triangle ABC$ upon $\triangle FGH$ in such a way that AB coincides with its equal FG, and so that point A falls on F and point B on G.	1. § 20. Postulate II. Any geometric figure may be moved from one place to another without changing its size or shape.
2. AC will fall on FH.	2. $\angle A$ is given equal to $\angle F$.
3. Point C will fall on point H.	3. AC is given equal to FH.
4. Therefore, since B falls on G and C on H, BC will coincide with GH.	4. § 19. Postulate I. There is only one straight line through two points.
5. $\triangle ABC$ is congruent to $\triangle FGH$.	5. § 24. The boundaries of the triangles coincide.

26. Corresponding parts. If two triangles have the three angles of one equal respectively to the three angles of the other, any two equal angles (one chosen from each triangle) are called *corresponding* or *homologous* angles, and any two

sides which lie opposite equal angles are called corresponding or homologous sides.

Thus, if $\angle A = \angle Y$, $\angle B = \angle X$, and $\angle C = \angle Z$, then $\angle A$ and $\angle Y$, $\angle B$ and $\angle X$, and $\angle C$ and $\angle Z$ are corresponding angles. Further, BC and XZ, AC and YZ, AB and YX, are corresponding sides.



Also, if any two rectilinear figures are congruent, any angle or side of one and the equal and similarly placed angle or side of the other are called corresponding parts.

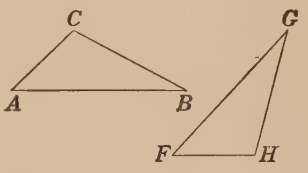
27. Parts of congruent figures. From § 26 it follows also that

Corresponding parts of congruent figures are equal.

It should be noted that in plane geometry the word *part* refers either to lines or angles.

Exercises

1. Suppose $\triangle ABC$ congruent to $\triangle FGH$. Name the other corresponding parts if (*a*) $\angle A = \angle F$ and $\angle B = \angle G$; (*b*) $\angle A = \angle F$, $AC = FH$, and $AB = FG$; (*c*) $AB = FG$, $\angle A = \angle F$, and $\angle B = \angle G$; (*d*) $AB = FG$ and $AC = FH$.



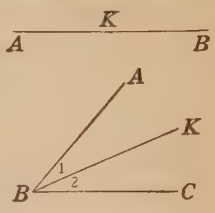
2. In answering (*a*) to (*d*) the student may have been guided by his eye and not by a real understanding of the definition of § 26. To make it emphatic that *corresponding parts are similarly situated parts*, we shall now assume that $\triangle ABC$ and FGH are congruent, and shall make some suppositions which the shape of the triangles really contradicts.

Name two corresponding parts if (*a*) $\angle A = \angle F$ and $\angle C = \angle G$; (*b*) $\angle B = \angle H$ and $\angle C = \angle G$; (*c*) $AC = FG$ and $AB = FH$.

Query 1. If two line-segments are equal, are they congruent?

Query 2. If two angles are equal, are they congruent?

28. Bisection. To *bisect* a magnitude means to divide it into two equal parts.



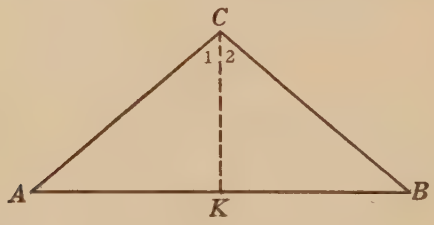
Thus, if AK equals KB, the point K bisects the line-segment AB.

Also, if $\angle 1 = \angle 2$, the line BK bisects the $\angle ABC$.

For the present we assume as evident that any line-segment or angle can be bisected. Later it will be shown how to perform these constructions

Theorem 2

29. *If two sides of a triangle are equal, the angles opposite them are equal.*



Given the triangle *ABC*, in which *AC* equals *BC*.

To prove *that* $\angle A = \angle B$.

Proof

Statement	Reason
1. Let CK bisect $\angle ACB$ and meet AB at some point K.	1. § 28. Any angle may be bisected.
2. Then in $\triangle ACK$ and BCK, $\angle 1 = \angle 2$.	2. Construction.
3. $AC = BC$.	3. Given.
4. $CK = CK$.	4. Identical.

5. $\triangle AOK$ is congruent to $\triangle BOK$.		5. § 25. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the two triangles are congruent.
6.	$\angle A = \angle B$.	6. § 27. Corresponding parts of congruent figures are equal.

NOTE. It should be observed as an implicit assumption of plane geometry that any lines or points added to a figure are in the same plane as the figure.

30. Isosceles triangle. An *isosceles triangle* is a triangle which has two equal sides.

EXERCISES

3. State Theorem 2, using the word *isosceles*.

4. If the three sides of a triangle are equal, its three angles are equal.

HINTS. Prove by using Theorem 2 twice.

Write down the several steps of the proof as in the proof of Theorems 1 and 2 above.

31. ***Axiom I.*** *If equals are added to equals, the results are equal.*

32. ***Axiom II.*** (1) *Two numbers or magnitudes each equal to a third are equal to each other.*

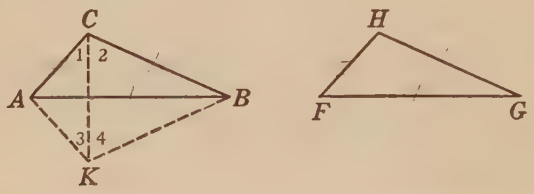
This axiom is often needed in the following form:

(2) *Two figures congruent to a third are congruent to each other.*

One should quote the form (1) or (2) of Axiom II as the case requires.

Theorem 3

33. *If the three sides of one triangle are equal respectively to the three sides of another, the triangles are congruent.*



Given the triangles *ABC* and *FGH* in which *AB* equals *FG*, *BC* equals *GH*, and *AC* equals *FH*.

To prove *that* $\triangle ABC$ *is congruent to* $\triangle FGH$.

Proof

Let AB and FG be the longest sides of the triangles respectively.

1. Suppose $\triangle FGH$ is placed adjacent to $\triangle ABC$ so that FG coincides with its equal AB, the point F falling on the point A, the point G on the point B, and the point H at K.	1. § 20. Postulate II. Any figure may be moved from one place to another without changing its size or shape.
2. Draw CK forming angles 1 and 2 at C and 3 and 4 at K.	2. § 19. Postulate I. There is only one straight line through two points.
3. $AC = AK$.	3. Given.
4. Therefore $\angle 1 = \angle 3$.	4. § 29. If two sides of a triangle are equal, the angles opposite them are equal.
5. Also $BC = BK$.	5. Given.
6. And $\angle 2 = \angle 4$.	6. § 29.

7. $\angle 1+\angle 2=\angle 3+\angle 4$ or $\angle ACB=\angle AKB$.	7. § 31. Axiom I. If equals are added to equals, the results are equal.
8. $\triangle ABC$ is congruent to $\triangle ABK$.	8. § 25. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the two triangles are congruent.
9. Then $\triangle ABC$ is congruent to $\triangle FGH$.	9. § 32. Axiom II. Two figures congruent to a third are congruent to each other.

Query 1. Does the proof of Theorem 3 depend on Theorem 2? on Theorem 1? If the proof of a theorem depends on Theorem 2, does it depend on Theorem 1?

Query 2. If three angles of one triangle are equal respectively to three angles of another, are the triangles congruent? Illustrate.

Query 3. Draw a figure for Theorem 3 assuming AB and FG the shortest sides respectively of the triangles. Would any difficulty then arise in the proof?

Exercises

5. In a triangle ABC line CK is drawn from C to the middle point of AB. If AC and BC are the same length, prove that $\triangle ACK$ is congruent to $\triangle BCK$.

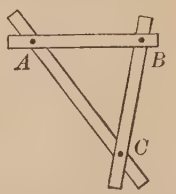
Hint. Show that Theorem 3 applies.

6. In a certain rectilinear four-sided figure the opposite sides are equal. Prove that a line joining two opposite vertices forms two equal triangles.

Hint. Use § 33.

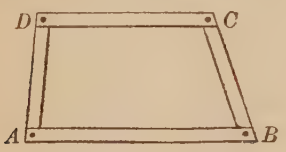
7. $ABCDE$ is a rectilinear figure having equal sides and equal angles. Prove that AD equals AC.

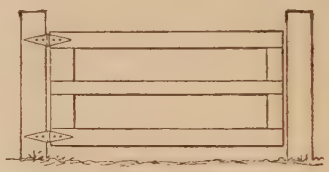
Hint. Apply Theorem 1, then § 27.



Query 1. Will three bars held together by one bolt at A, B, and C respectively form a rigid figure?

QUERY 2. Will a quadrilateral formed by four bars and held together by one bolt at A, B, C, and D respectively form a rigid figure? How can it be made rigid?





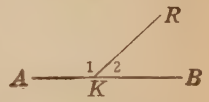
QUERY 3. Will the gate (of the above figure) sag from its own weight? If so, how can sagging be prevented?

34. Adjacent angles. Two angles which have the same vertex and a common side between them are called *adjacent angles.*

Thus angle ABK and angle KBC are adjacent angles.

35. Straight angle. A *straight angle* is an angle whose sides extend in opposite directions and form a straight line.

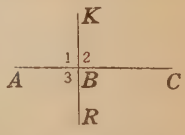
In the adjacent figures, if AKB is a straight line, $\angle 1 + \angle 2 =$ a straight angle and $\angle 3 + \angle 4 + \angle 5 =$ a straight angle.



36. ***Postulate III.*** *All straight angles are equal.*

It follows that if two lines cross so that any two of the adjacent angles are equal, all four angles formed are equal, because each is half a straight angle. Thus, if in the figure of § 37 AC and KR cross so that $\angle 1 = \angle 3$, then $\angle 1 = \angle 2 = \angle 3 = \angle CBR$.

37. Perpendicular. If one straight line cuts another so as to make any two adjacent angles equal, each line is *perpendicular* (symbol $\perp$) to the other.



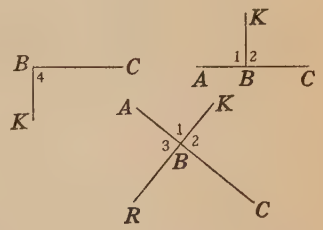
Thus, if ABC and KBR are straight lines, and angle ABK equals angle KBC, each of the lines AC and KR (or any portion of either) is perpendicular to the other.

If one straight line cuts another so that there are no two adjacent angles which are equal to each other, then the lines would not be called perpendicular.

A corresponding remark holds regarding most of the definitions in this and other texts.

38. Right angle. If one line is perpendicular to another, the four angles between them are called *right angles.*

Thus, in the adjacent figures, if KB is $\perp$ to AC, AC to BR, and BC to BK respectively, the angles 1, 2, 3, and 4 are right angles.



39. ***Axiom III.*** *If equals are divided by the same number, the results are equal.*

40. Straight angle and right angle. From the definition of a right angle it follows that a right angle is half a straight angle, or a straight angle is twice as great as a right angle.

Hence the sum of the angles around a fixed point is four right angles.

From § 36 and § 39 it immediately follows that

All right angles are equal.

41. In the figure of § 37, if BK is perpendicular to AC, no other line in the plane of the paper can be drawn through B perpendicular to AC. This gives

Postulate IV. *At a given point of a line, one and only one perpendicular can be drawn to the line.*

Thus, if KR is perpendicular to AB, it bisects the straight angle AKB. Obviously a second $\perp$ KL is impossible, because an angle cannot be bisected by two different lines.

QUERY 1. In the figure of § 34, do the angles ABK and ABC have the same vertex and a common side? Are they adjacent angles?

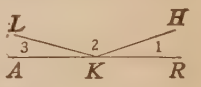
QUERY 2. In the figure of § 37, if KB cuts AC, how many right angles are formed?

QUERY 3. Can two angles have a common side and not be adjacent angles? Illustrate with a figure.

QUERY 4. Can two angles have a common vertex and not be adjacent angles? Illustrate.

QUERY 5. If two perpendicular lines intersect, how many right angles are formed?

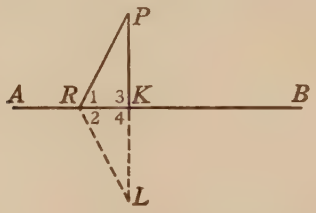
QUERY 6. If one line is perpendicular to a second, is the second perpendicular to the first?



QUERY 7. If LKR is not a straight line, can the sum of the angles 1 and 2 equal two right angles? If AKR is a straight line, what is the sum of $\angle 1 + \angle 2$? $\angle 1 + \angle 2 + \angle 3$?

Theorem 4

42. *There is only one perpendicular from a point to a line.*



Given the point *P* any point outside the line *AB*, *PK* a perpendicular to *AB* at *K*, and *PR* any other line from *P* terminating in *AB*.

To prove that PR *is not* $\perp$ *to* AB.

Proof

1. Let PK be extended to L, making $KL = PK$, and draw LR.	1. § 11. A straight line extends indefinitely.
2. Now in $\triangle PKR$ and LKR $\angle 3 = \angle 4$.	2. § 40. All right angles are equal.
3. $PK = LK$.	3. Construction.

4. $KR = KR$.	4. Identical.
5. $\triangle PKR$ is congruent to $\triangle LKR$.	5. § 25. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the two triangles are congruent.
6. $\angle 1 = \angle 2$.	6. § 27. Corresponding parts of congruent figures are equal.
7. Since PKL is a straight line, PRL is not a straight line.	7. § 19, Postulate I. There is only one straight line through two points.
8. $\angle 1 + \angle 2$, or $\angle PRL$, is not a straight angle.	8. § 35. A straight angle is an angle whose sides extend in opposite directions and form a straight line.
9. $\angle 1$, or half of $\angle PRL$, is not a right angle.	9. § 40. A right angle is half a straight angle.
10. PR is not $\perp$ to AB.	10. Step 9. § 38. If one line is perpendicular to another, the four angles between them are called right angles.

NOTE. If A, B, and C are three theorems, and A is quoted to prove B but B alone is quoted to prove C, then C depends *directly* upon B and *indirectly* upon A.

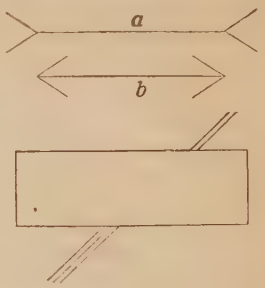
QUERY. Does the proof of Theorem 4 depend directly or indirectly on Theorem 3? on Theorem 2? on Theorem 1?

EXERCISES

8. Prove that two of the three angles of a triangle cannot both be right angles.

NOTE. When unable to prove some property of a figure, one is tempted to say, "Why, you can *see* that it is true." The following illustrations, however, will show that the eye may easily be deceived, and that an appeal to the eye is not a proof.

9. Which appears to be the longer in the adjacent figure, a or b? Measure each. Which is the longer?



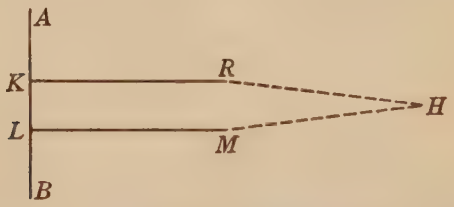
10. Draw freehand a perpendicular to AB at its mid-point and equal to AB. Measure the two lines. Are they equal?

11. Which lines of the adjacent figure are prolongations of the others? Verify your answer.

43. Parallel lines. *Parallel lines* are lines that lie in the same plane and do not meet however far they are produced.

Theorem 5

44. *Two lines perpendicular to the same line are parallel.*



Given the lines KR and LM each perpendicular to AB at K and L respectively.

To prove that KR *is* $\parallel$ *to* LM.

Proof

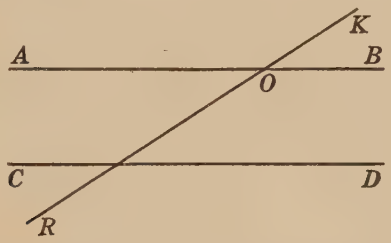
1. Since KR and LM lie in the same plane, they either meet or they do not meet.	1. No other possibility exists.
2. If they meet, as at H, there will be two perpendiculars from that point to AB.	2. Hypothesis.

3. But two perpendiculars from the same point to a line are impossible.	3. § 42. There is only one perpendicular from a point to a line.
4. Therefore KR cannot meet LM.	4. Remaining alternative in statement 1.
5. Therefore KR is $\parallel$ to LM.	5. § 43.

45. *Postulate V.* The postulate of parallels. *Through a given point outside a line, one line parallel to it exists, and only one.*

Theorem 6

46. *If a line intersects one of two parallel lines, it intersects the other also.*



Given the parallels *AB* and *CD* and the line *KR* cutting *AB* at *O*.

To prove *that* *KR cuts CD.*

Proof

1. KR intersects CD or it does not intersect it.	1. No other possibility exists.
2. If it does not, then KR and CD are $\parallel$.	2. § 43.
3. Then AB and KR pass through O and are $\parallel$ to CD. But this is impossible.	3. § 45.
4. Then if KR is not $\parallel$ to CD, it intersects it.	4. Remaining alternative in statement 1.

NOTE. The method of proof in which we suppose one geometric figure placed upon another, as was done in Theorem 1, is called a proof by *superposition*.

QUERY 1. Is any theorem from 2 to 6 proved by superposition?

QUERY 2. On what postulate must every proof by superposition depend?

QUERY 3. On what theorems from 1 to 4 does the proof of Theorem 5 directly depend? indirectly depend?

QUERY 4. On which of the five preceding theorems does the proof of Theorem 6 directly depend? indirectly depend?

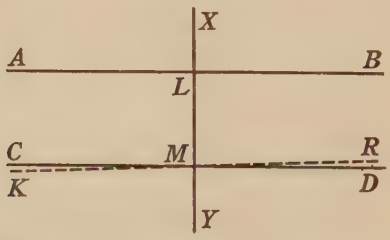
QUERY 5. Can three pencils be held so that two of them are perpendicular to the third at the same point? Illustrate.

QUERY 6. Can two pencils be held so that they are perpendicular to the same line at different points, but are not parallel to each other? Illustrate.

NOTE. The answer to the preceding query illustrates the fact that in the statements of the theorems of plane geometry it is implicitly understood that all the points and lines referred to by the theorem lie in one plane. This is important, for without this understanding Theorem 5 as stated is not true.

Theorem 7

47. *If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*



Given the parallel lines *AB* and *CD*, with *XY* perpendicular to *AB* and intersecting it at *L*.

To prove that ... *XY is* $\perp$ *to CD.*

Proof

1. XY intersects CD at some point, as M.	1. § 46. If a line intersects one of two parallel lines, it intersects the other also.
2. CD is $\perp$ to XY at M or it is not $\perp$ to XY.	2. No other possibility exists.
3. Suppose CD is not $\perp$ to XY. And let KR be drawn through $M \perp$ to XY.	3. § 41.
4. Then KR is $\parallel$ to AB.	4. § 44. Two lines perpendicular to the same line are parallel.
5. Since KR and CD are $\parallel$ to AB and both pass through point M, they make but one line, instead of being two distinct lines as indicated in the figure. This means that KR, when drawn through $M \perp$ to XY, coincides with CD.	5. § 45. Postulate V. Through a given point outside a line, one line parallel to it exists, and only one.
6. KR is $\perp$ to XY.	6. Construction.
7. CD is $\perp$ to XY.	7. Since CD and KR are the same line.

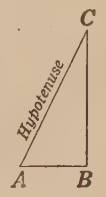
QUERY 1. On which of the preceding theorems does the proof of Theorem 7 directly depend? indirectly depend?

QUERY 2. Can you hold three pencils so that two are parallel and the third is perpendicular to one of them but not to the other?

QUERY 3. How many of the theorems 1 to 7 are true only when the figure is supposed to lie in a plane?

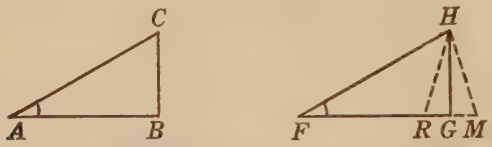
48. Right triangle. A *right triangle* is a triangle in which one angle is a right angle.

49. Hypotenuse. The *hypotenuse* of a right triangle is the side opposite the right angle.



Theorem 8

50. *Two right triangles are congruent if the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.*



Given the right triangles *ABC* and *FGH* in which the hypotenuse *AC* equals the hypotenuse *FH* and the angle *A* equals the angle *F*.

To prove *that* $\triangle ABC$ *is congruent to* $\triangle FGH$.

Proof

1. Place $\triangle ABC$ upon $\triangle FGH$ so that the hypotenuse AC coincides with its equal FH, and so that the point A falls on the point F and the point C on the point H.	1. § 20.
2. Now $\angle A = \angle F$.	2. Given.
3. Hence AB will fall along FG.	3. Statements 1 and 2.
4. CB must fall in one of three positions: upon HG, to the left as HR, or to the right as HM.	4. No other possibility exists.
5. If CB falls in the position HR, then HR and HG would form two $\perp$s from H to FG, which is impossible.	5. § 42. There is only one perpendicular from a point to a line.

In like manner it can be proved that CB cannot fall on HM.

6. Therefore CB falls on HG.	6. Remaining alternative in 4.
7. Therefore $\triangle ABC$ is congruent to $\triangle FGH$.	7. § 24.

51. ***Axiom IV.*** *If equals are subtracted from equals, the results are equal.*

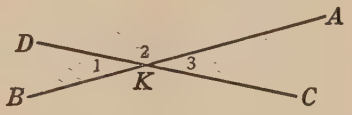
52. Vertical angles. Two angles are *vertical angles* if the sides of one are the prolongations of the sides of the other.

QUERY 1. In which of the first eight theorems did the proof depend on no other one of the eight?

QUERY 2. If two straight lines intersect, how many pairs of vertical angles are formed? how many pairs of adjacent angles?

Theorem 9

53. *If two straight lines intersect, the vertical angles are equal.*



Given the straight lines *BA* and *DC* intersecting at *K*.

To prove that $\angle 1 = \angle 3$ *and* $\angle 2 = \angle BKC$.

Proof

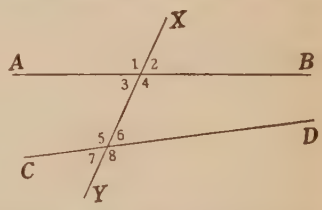
1. $\angle 1 + \angle 2 =$ a straight angle.	1. § 35.
2. $\angle 2 + \angle 3 =$ a straight angle.	2. § 35.
3. $\angle 1 + \angle 2 = \angle 2 + \angle 3$.	3. § 36. All straight angles are equal.
4. $\angle 1 = \angle 3$.	4. § 51. If equals are subtracted from equals, the results are equal.
5. Again $\angle 1 + \angle 2 =$ a straight angle.	5. Why?
6. And $\angle 1 + \angle BKC =$ a straight angle.	6. Why?
7. Hence $\angle 2 = \angle BKC$.	7. Why?

Exercise 12. Two straight lines KRL and GRH intersect so that KR equals RH and GR equals RL. Draw KG and HL and prove that the triangle KRG is congruent to the triangle HRL.

Query. In the figure of the preceding exercise, if the lines KH and GL be drawn, are the triangles KRH and GRL congruent? Why?

54. Transversal. A *transversal* is a line that crosses (cuts or intersects) two or more lines.

55. Angles made by a transversal. In the adjacent figure the angles 1, 2, 7, and 8 are called *exterior* angles and angles 3, 4, 5, and 6 are called *interior* angles.



The two pairs of angles 3 and 6, 4 and 5, are called *alternate-interior* angles; and the two pairs of angles 1 and 8, 2 and 7, are called *alternate-exterior* angles.

The four pairs of angles 1 and 5, 2 and 6, 3 and 7, and 4 and 8 are called *corresponding* angles.

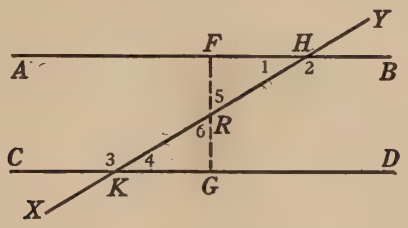
It should be noted that we now have applied the word *corresponding* to certain angles of congruent figures and to certain angles formed when a transversal cuts two other lines, which will almost invariably be two parallel lines. Therefore, in using the word *corresponding* we should always be careful to convey the meaning we intend, by saying either corresponding angles of *parallel* lines or corresponding angles of *congruent* figures.

56. Degree. If a right angle be divided into ninety equal angles, each part is an angle of one *degree*.

Exercise 13. Draw a figure like that of § 55, making $\angle 6$ about 40° and AB parallel to CD. How many angles appear to contain 40°? How many to contain 140°? How many still another number of degrees?

Theorem 10

57. *If two parallel lines are cut by a transversal, the alternate-interior angles are equal.*



Given the two parallels *AB* and *CD* cut by the transversal *XY* at *H* and *K* respectively, forming the pairs of alternate-interior angles 1 and 4, 2 and 3.

To prove that $\angle 1 = \angle 4$ *and* $\angle 2 = \angle 3$.

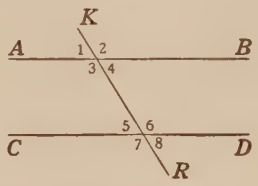
Proof

1. Let R be the mid-point of HK, and let the line through R which is $\perp$ to AB meet AB in F and CD in G. Then FG is $\perp$ to CD.	1. § 47. If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.
2. $\angle 5 = \angle 6$.	2. § 53. If two straight lines intersect, the vertical angles are equal.
3. $\triangle KRG$ is congruent to $\triangle FRH$.	3. § 50.
4. $\angle 1 = \angle 4$. (1)	4. § 27. Corresponding parts of congruent figures are equal.
5. But $\angle 1 + \angle 2 = \angle 3 + \angle 4$. (2)	5. § 36. All straight angles are equal.
6. $\angle 2 = \angle 3$.	6. (2)−(1), § 51. If equals are subtracted from equals, the results are equal.

58. Corollary. A *corollary* is a secondary, or a minor, theorem dependent on a major theorem. Usually it is a theorem the truth of which is apparent from the major theorem or which follows from it with but little proof.

The theorems of §§ 59 and 60 are corollaries to Theorem 10. The proof of each is brief and depends on Theorem 10.

59. *Corollary 1.* *If two parallel lines are cut by a transversal, the corresponding angles are equal.*



Given the parallels *AB* and *CD* cut by the transversal *KR*, forming four pairs of corresponding angles 1, 5; 2, 6; 3, 7; 4, 8.

To prove that $\angle 1 = \angle 5$.

HINTS.

$\angle 1 = \angle 4$.	Why?
$\angle 4 = \angle 5$.	Why?
$\therefore \angle 1 = \angle 5$.	Why?

EXERCISES

14. Prove in like manner (using the figure of § 59) that $\angle 2 = \angle 6$.

15. Prove in like manner that $\angle 3 = \angle 7$.

16. Prove in like manner that $\angle 4 = \angle 8$.

17. If two parallel lines are cut by a transversal, the alternate-exterior angles are equal.

Prove, using the figure of § 59, that $\angle 1 = \angle 8$ and $\angle 2 = \angle 7$.

HINTS.

$\angle 1 = \angle 4$.	Why?
$\angle 4 = \angle 5$.	Why?
$\angle 5 = \angle 8$.	Why?
$\therefore \angle 1 = \angle 8$.	Why?

In like manner it can be proved that $\angle 2 = \angle 7$.

NOTE. In some of the proofs of the preceding theorems and corollaries the word "Why?" has been placed on the right of the page. Its presence denotes the omission of a reason (see § 22) at that point in the demonstration. The purpose of the omission is to lead the student to supply mentally the reason omitted and thus to do some of his own thinking even in the proofs given in the text. He should in all such cases pause long enough to decide in his own mind what the omitted reason is. In oral or written demonstrations of these theorems and corollaries all necessary reasons should be given in full, whether they are given in the text or not.

60. ***Corollary 2.*** *If two parallel lines are cut by a transversal, the sum of the two interior angles on the same side of the transversal equals two right angles.*

Given the parallel lines *AB* and *CD* cut by the transversal *KR*.

To prove, *using the figure of § 59, that*

$$\angle 3 + \angle 5 = 2 \text{ rt. } \measuredangle, \text{ and } \angle 4 + \angle 6 = 2 \text{ rt. } \measuredangle.$$

HINTS. $\angle 3 + \angle 4 = 2$ rt. $\measuredangle$. (1) Why?

$\angle 4 = \angle 5$. Why?

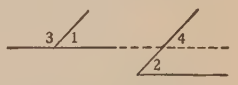
Substituting $\angle 5$ for $\angle 4$ in (1), $\angle 3 + \angle 5 = 2$ rt. $\measuredangle$.

In like manner it can be proved that

$$\angle 4 + \angle 6 = 2 \text{ rt. } \measuredangle.$$

61. Supplementary angles. One angle is the *supplement* of another if their sum equals two right angles (or one hundred and eighty degrees).

62. ***Corollary 3.*** *If two angles have their sides parallel each to each, they are equal or supplementary.*



HINTS. What is the relation between $\angle 1$ and $\angle 4$? $\angle 4$ and $\angle 2$? What, then, is the relation between $\angle 1$ and $\angle 2$? What is the relation between $\angle 3$ and $\angle 1$? What, then, is the relation between $\angle 3$ and $\angle 2$?

63. Converse. In the statement of every theorem there is a conditional part (often expressed in an "if" clause) called the *hypothesis.* The other part of the theorem is a consequence of the hypothesis and is called the *conclusion.* In any theorem, if the hypothesis and the conclusion be interchanged, the resulting theorem is called the *converse* of the first.

For example, the converse of Theorem 1 is as follows: If two triangles are congruent, two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

Exercises

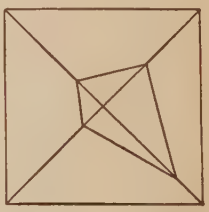
18. State the hypotheses of Theorems 1 to 10 inclusive. State the conclusions.

19. State the converse of Theorems 2, 3, and 10 respectively.

20. State the converse of the corollaries in §§ 59, 60, and 62 respectively.

It is important to note that the converse of a theorem is not always true.

Thus: If a quadrilateral is a square, its diagonals intersect at right angles. Conversely: If the diagonals of a quadrilateral intersect at right angles, the quadrilateral is a square. From the larger figure on the right the first theorem appears to be true, while the smaller quadrilateral shows that the converse is false. Even outside of geometry the converse of a theorem is not always true. This will be apparent from stating the converse of the following: If a man lives in Chicago, he lives in Illinois. These illustrations make it clear that though a theorem is true, its converse may not be true. Until the converse of a theorem has been proved it must not be quoted as a reason for any statement.



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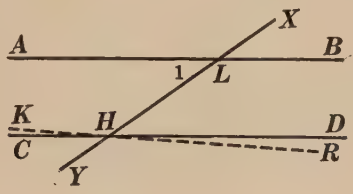
Theorem 11 *(Converse of Theorem 10)*

64. *If two straight lines are cut by a transversal, making two alternate-interior angles equal, the lines are parallel.*



Given *AB* and *CD* cut by the transversal *XY* at *L* and *H* respectively, making the alternate-interior angles 1 and *LHD* equal.

To prove that *AB is ∥ to CD.*

Proof

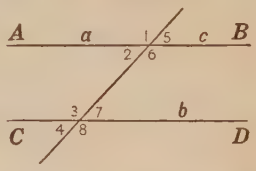
1. *CD* is ∥ to *AB* or it is not ∥ to *AB*.	1. No other possibility exists.
2. Suppose *CD* is not ∥ to *AB*. Draw *KR* through *H* ∥ to *AB*.	2. § 45.
3. Then $\angle 1 = \angle LHR$.	3. § 57. If two parallel lines are cut by a transversal, the alternate-interior angles are equal.
4. $\angle 1 = \angle LHD$.	4. Hypothesis.
5. $\angle LHR = \angle LHD$.	5. § 32.
6. Hence *KHR* and *CHD* are not two distinct lines, but the same line.	6. § 24.
7. But *KHR*, or *KR*, is ∥ to *AB*.	7. Statement 2.
8. Therefore *CHD*, or *CD*, which coincides with *KR*, is ∥ to *AB*.	8. § 45.

Exercises

21. If, in the adjacent figure, $\angle 1 = \angle 8$, prove that AB is $\parallel$ to CD.

22. If $\angle 5 = \angle 7$, prove that AB is $\parallel$ to CD.

23. If $\angle 2 + \angle 3 = 2$ rt. $\measuredangle$, prove that AB is $\parallel$ to CD.



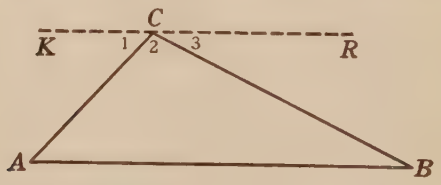
24. From the preceding exercises state three corollaries to Theorem 11.

65. ***Axiom V.*** *A number may be substituted for its equal in any operation on numbers.*

Note. It should be observed that the axioms stated from time to time apply to numbers. If, then, for example, we say subtract line AB from line CD or substitute $\angle x$ for $\angle a$, it is the numerical measure of the line or the angle respectively to which we refer and to which the axiom applies.

Theorem 12

66. *The sum of the angles of any triangle is two right angles.*



Given the triangle *ABC*.

To prove that $\angle A + \angle B + \angle C = 2$ *rt.* $\measuredangle$.

Proof

1. Let KR be a line through $C \parallel$ to AB, forming $\angle 1$ and $\angle 3$ respectively.	1. § 45.
2. Then $\angle 1 + \angle 2 + \angle 3 =$ st. $\angle KCR = 2$ rt. $\measuredangle$. (1)	2. §§ 35, 40.

3. $\angle 1 = \angle A$.	3. § 57. If two parallel lines are cut by a transversal, the alternate-interior angles are equal.
4. Also $\angle 3 = \angle B$.	4. Why?
5. $\angle A + \angle 2 + \angle B = 2$ rt. $\measuredangle$s, or $\angle A + \angle B + \angle C = 2$ rt. $\measuredangle$s.	5. Substituting $\angle A$ for $\angle 1$ and $\angle B$ for $\angle 3$ in equation (1), § 65.

67. ***Corollary 1.*** *If two angles of one triangle equal respectively two angles of another, the third angle of the first equals the third angle of the second.*

Given two triangles whose angles are denoted by a, b, c, and x, y, z respectively, where $a = x$ and $b = y$.

To prove that $c = z$.

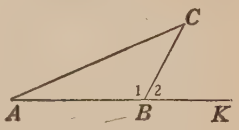
Proof

1. Now $a + b + c = x + y + z = 2$ rt. $\measuredangle$s. (1)	1. § 66.
2. $a + b = x + y$. (2)	2. Hypothesis and § 31.
3. From (1) and (2), $c = z$.	3. § 51.

68. Exterior angle. An *exterior* angle of a triangle is the angle between any side and the adjacent side produced. See $\angle 2$ of § 69.

QUERY. How many exterior angles has a triangle?

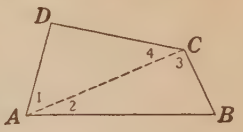
69. ***Corollary 2.*** *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*



HINTS.	$\angle 1 + \angle 2 = 2$ rt. $\measuredangle$s.	Why?
	$\angle 1 + \angle A + \angle C = 2$ rt. $\measuredangle$s.	Why?
Hence	$\angle 1 + \angle 2 = \angle 1 + \angle A + \angle C$.	Why?
Therefore	$\angle 2 = \angle A + \angle C$.	Why?

70. Quadrilateral. A *quadrilateral* is a portion of a plane bounded by four straight lines.

71. ***Corollary 3.*** *The sum of the angles of a quadrilateral is four right angles.*



Proof. Let $ABCD$ be any quadrilateral. Draw AC.

1. Then $\angle 1+\angle 4+\angle D=2$ rt. $\measuredangle$. Why?
2. And $\angle 2+\angle 3+\angle B=2$ rt. $\measuredangle$. Why?
3. $(\angle 1+\angle 2)+(\angle 4+\angle 3)+(\angle B+\angle D)=4$ rt. $\measuredangle$. Why?

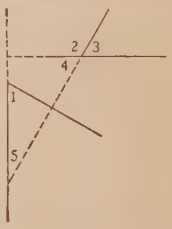
72. Complementary angles. One angle is the *complement* of another if their sum is a right angle (or ninety degrees).

73. Acute angle. An *acute* angle is an angle less than a right angle.

74. Obtuse angle. An *obtuse* angle is an angle greater than a right angle and less than two right angles.

75. ***Corollary 4.*** *The two acute angles of a right triangle are complementary.*

76. ***Corollary 5.*** *If two angles have their sides perpendicular each to each, they are equal or supplementary.*



HINTS. Let the sides of $\angle 1$ be $\perp$ respectively to the sides of $\angle 3$. What is the relation between $\angle 3$ and $\angle 4$? $\angle 4$ and $\angle 5$? $\angle 5$ and $\angle 1$? Conclusion?

What is the relation between $\angle 2$ and $\angle 3$? $\angle 2$ and $\angle 1$?

77. Conjugate angles. One angle is the *conjugate* of another if their sum is four right angles.

78. Polygon. A *polygon* is a plane figure bounded by three or more straight lines.

79. Regular polygon. A *regular polygon* is a polygon all of whose angles are equal and all of whose sides are equal.

Such a polygon is said to be equiangular and equilateral.

80. Diagonal. A *diagonal* of a polygon is a line joining any two nonconsecutive vertices.

81. Special polygons. A *pentagon* is a polygon having five sides. A *hexagon* is a polygon having six sides.

Algebraic Exercises on Angle Relations

25. Solve for $\angle x$, $20° + \angle x = 38° 30'$.

26. Solve for $\angle x$, $3\angle x + 10° = 64° + \angle x$.

27. Solve for x, $180° - x = \frac{2}{3}(90° - x)$.

28. What is the complement of 37°? the supplement?

29. What is the conjugate of 130°?

30. One acute angle of a right triangle is four times the other. Find the number of degrees in each.

Hint. Let $4x$ and x represent the number of degrees in the two angles respectively.

31. Two parallels are cut by a transversal. One of the two interior angles on the same side of the transversal is five times the other. Find the number of degrees in each.

32. In a certain quadrilateral whose opposite sides are parallel, one of two adjacent angles is four times the other. Find the number of degrees in each angle.

33. If x represents the number of degrees in one angle, represent its complement; its supplement; its conjugate.

34. The complement of an angle is three and one-half times the angle. Find the angle.

35. The supplement of an angle is 21° more than twice the angle. Find the angle.

36. One angle of a triangle is twice another, and the third is 15° more than four times the sum of the other two. Find each.

37. The complement of an angle is 52° less than two thirds the supplement of the angle. Find the angle.

38. In the triangle ABC, the angle B is twice the angle A. The bisectors of these angles meet at D, and BD produced meets AC at R. The angle BRC equals 72°. Find the angles of the triangle ABC.

39. The side AB of the triangle ABC is extended to K. The bisectors of the angles CBK and A meet at R. Assign a numerical value to each angle of the triangle ABC. Then determine the number of degrees in the angle R. Compare this with the angle C.

40. Assign other values to the angles of the triangle ABC and repeat the work of the preceding exercise.

41. Find the sum of the angles of a pentagon.

HINT. From one vertex draw all the diagonals possible.

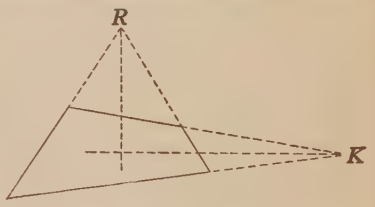
42. Find each angle of an equiangular pentagon.

43. Find the sum of the angles of an equiangular hexagon.

44. If a pentagon is regular, find the number of degrees in the angle between two diagonals drawn from the same vertex.

45. A certain hexagon is equilateral and equiangular. Find the angle between each adjacent pair of diagonals.

46. Assign unequal numerical values to the angles of a quadrilateral, making the opposite angles supplementary, and produce the opposite sides of the quadrilateral until they meet in K and R. Then bisect the angles K and R and determine the number of degrees in the angle between the bisectors.



47. Solve the preceding exercise again, assigning other values to the angles of the quadrilateral than those used before. What is the angle between the bisectors? Is a conclusion possible here?

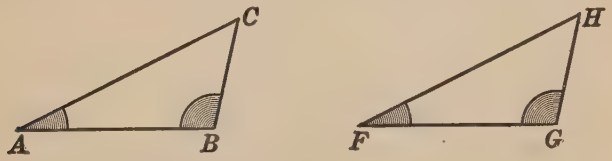
48. $ABCD$ is a quadrilateral with its diagonals AC and BD intersecting at K. The bisectors of the angle ADB and the angle ACB meet in R. Assign numerical values to the angle ADB, the

angle ACB, and the angle AKD. Then determine the number of degrees in the angle DRC.

49. Solve the preceding exercise again, assigning still other values to the angles. Is the relation between the angle DRC and the sum of the angles DAC and DBC the same as before? Is a conclusion from this fact possible?

Theorem 13

82. *If a side and the two adjacent angles of one triangle are equal respectively to a side and the two adjacent angles of another, the triangles are congruent.*



Given the triangles *ABC* and *FGH* in which the side *AB* equals the side *FG*, the angle *A* equals the angle *F*, and the angle *B* equals the angle *G*.

To prove *that* $\triangle ABC$ *is congruent to* $\triangle FGH$.

Proof

1. Superpose $\triangle FGH$ on $\triangle ABC$ so that FG falls on its equal AB, the point F falling on the point A and the point G on the point B.	1. § 20.
2. The side FH will fall on AC.	2. $\angle A = \angle F$.
3. The side GH will fall on BC.	3. $\angle G = \angle B$.
4. The point H falls at the intersection C of the lines AC and BC.	4. It falls on the line AC and also on the line BC.
5. Therefore $\triangle ABC$ is congruent to $\triangle FGH$.	5. § 24.

Exercises

Solve for x and y and check:

50. $x - 2y = 8,$
$3x + 2y = 7.$

51. $5x - 3y = 2,$
$15x + 12y = -5.$

52. Solve for $\angle A, \angle B,$ and $\angle C$:
$\angle A + \angle B + \angle C = 180°,$
$\angle A + 2\angle B = 166°,$
$\angle A + \angle B - \angle C = 52°.$

53. The sum of two angles of a triangle is 105°. The sum of one of these and the third angle is 115°. Find the number of degrees in each angle of the triangle.

54. The difference of two angles of a triangle is 24°. The sum of one of these and the third angle is 132°. Find each angle of the triangle.

Historical Note. Theorem 13 is one of several theorems the discovery and proof of which are attributed to Thales of Miletus (about 600 B.C.), one of the Seven Wise Men of Greece. He applied this theorem to the measurement of the distances of ships from the shore. Business affairs took him for a time to Egypt, where he learned what the Egyptians knew of science and geometry, the study of which he later introduced into Greece. He seems to have been the first man to appreciate the necessity of a scientific proof in geometry. While yet in Egypt his acute mind enlarged on what he learned and, according to Plutarch, he soon excelled his Egyptian teachers and astonished King Amasis by measuring the heights of the pyramids from the lengths of their shadows. A more amazing performance, no doubt, was the prediction by Thales of an eclipse of the sun for the year 585 B.C. In fact, the scientific study of astronomy begins with him. It was Thales, also, who first noticed that a piece of amber when rubbed becomes electrified. Moreover, he expressed the conviction that there is some unifying principle which links together all the physical phenomena and is able to make them intelligible; and that all matter is made of one primordial element, the search for which should be the aim of natural science. Modern physics has shown that an intimate relation exists between heat, light, electricity, and magnetism, and it knows that electrons are a constituent of the atoms of all substances. Thus we see that, in addition to doing fundamental work in geometry, Thales was the first to describe the spirit which guided the growth of physical science in all

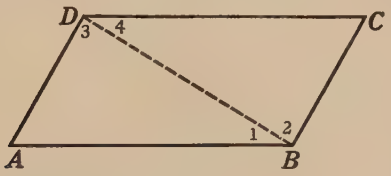
ages and to express his belief in the ultimate nature of matter, which the very latest work in physics comes near confirming. These facts constitute a very remarkable tribute to the genius of the man.

83. Parallelogram. A *parallelogram* is a quadrilateral whose opposite sides are parallel.

84. Rectangle. A *rectangle* is a parallelogram whose angles are right angles.

Theorem 14

85. *The opposite sides of a parallelogram are equal.*



Given the parallelogram *ABCD*.

To prove *that* $AB = DC$ *and* $AD = BC$.

Proof

1. Draw the diagonal BD.	1. § 19.
2. In $\triangle ABD$ and BCD, $\angle 2 = \angle 3$.	2. § 57. If two parallel lines are cut by a transversal, the alternate-interior angles are equal.
3. Also $\angle 1 = \angle 4$.	3. Why ?
4. $DB = DB$.	4. Why ?
5. $\triangle ABD$ is congruent to $\triangle BCD$.	5. § 82. If a side and the two adjacent angles of one triangle are equal respectively to a side and the two adjacent angles of another, the triangles are congruent.
6. Therefore $AB = DC$ and $AD = BC$.	6. § 27. Corresponding parts of congruent figures are equal.

86. ***Corollary.*** *The opposite angles of a parallelogram are equal.*

HINTS. In the figure of § 85,

$$\angle 2 = \angle 3, \qquad (1) \qquad \text{Why?}$$

and

$$\angle 1 = \angle 4. \qquad (2)$$

Therefore, from (1) and (2),

$$(\angle 1 + \angle 2), \text{ or } \angle ABC = (\angle 3 + \angle 4), \text{ or } \angle ADC. \quad \S\S 31 \text{ and } 65$$

Also $\angle A = \angle C$. Why?

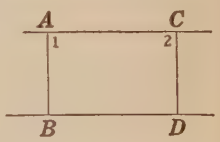
EXERCISES

55. A diagonal of a parallelogram divides it into two congruent triangles.

HINTS. In the proof of Theorem 14 it is shown that BD divides the parallelogram into two congruent triangles. It may be proved in like manner that AC does also.

56. If two lines are perpendicular to one of two parallels, the portions of them included between the two parallels are equal.

HINTS. Let A and C be any two points on one of the parallels. Let AB and CD be $\perp$ to the other parallel. $\angle 1 = ?$ Why? $\angle 2 = ?$ Why? Is $AB \parallel$ to CD? Why? What is $ABDC$? Why? Conclusion?



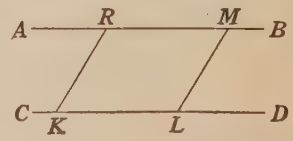
57. If one angle of a parallelogram is a right angle, the figure is a rectangle.

HINT. Use § 60 and § 84.

58. Parallel line-segments included between parallels are equal.

HINTS. Given $AB \parallel$ to CD, and KR and LM two other parallels included between them.

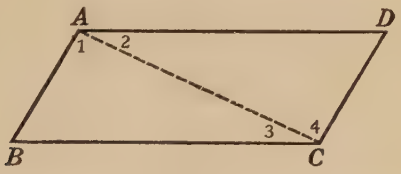
What kind of figure is $KLMR$? Why? What relation exists between KR and LM? Why?



59. If the angles of a quadrilateral are right angles, the figure is a rectangle.

Theorem 15 *(Converse of Theorem 14)*

87. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Given the quadrilateral *ABCD* in which *AB* equals *DC* and *AD* equals *BC*.

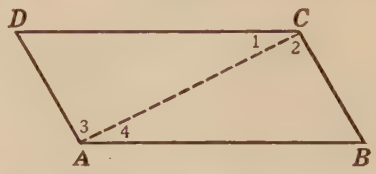
To prove *that* *ABCD is a parallelogram.*

Proof

1. Draw the diagonal AC.	1. § 19.
2. In the $\triangle ABC$ and ADC, $AB = DC$ and $AD = BC$.	2. Hypothesis.
3. $AC = AC$.	3. Identical.
4. $\triangle ABC$ is congruent to $\triangle ADC$.	4. § 33. If the three sides of one triangle are equal respectively to the three sides of another, the triangles are congruent.
5. $\angle 1 = \angle 4$.	5. § 27.
6. DC is $\parallel$ to AB.	6. § 64. If two straight lines are cut by a transversal, making two alternate-interior angles equal, the lines are parallel.
7. Also $\angle 3 = \angle 2$.	7. § 27.
8. AD is $\parallel$ to BC.	8. § 64.
9. From 6 and 8 it follows that $ABCD$ is a parallelogram.	9. § 83. A parallelogram is a quadrilateral whose opposite sides are parallel.

Theorem 16

88. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Given the quadrilateral *ABCD* with *AB* equal to *DC* and parallel to it.

To prove *that the quadrilateral ABCD is a parallelogram.*

Proof

1. Draw the diagonal AC.	1. § 19.
2. AB is $\parallel$ to DC.	2. Hypothesis.
3. Hence $\angle 1 = \angle 4$.	3. § 57. If two parallel lines are cut by a transversal, the alternate-interior angles are equal.
4. $AB = DC$.	4. Hypothesis.
5. $AC = AC$.	5. Identical.
6. $\triangle ABC$ is congruent to $\triangle CDA$.	6. § 25.
7. $\angle 2 = \angle 3$.	7. Why?
8. AD is $\parallel$ to BC.	8. § 64. If two straight lines are cut by a transversal, making two alternate-interior angles equal, the lines are parallel.
9. $ABCD$ is a parallelogram.	9. § 83.

89. Method of proof. Much of the student's work in the exercises and theorems of Book I really consists in proving two angles equal or two lines equal. Sometimes only

one theorem is needed to do this. Thus two angles can frequently be proved equal by pointing out that they are vertical angles, or that they are the base angles of an isosceles triangle, or that they are alternate-interior or corresponding angles of parallel lines; and two lines can often be proved equal by showing that they are the equal sides of an isosceles triangle, or that they are the opposite sides of a parallelogram. Occasionally a proof requires the use of only two theorems.

Among the various methods which require the use of three or more theorems there is one which the student must use so frequently that it will now be given special emphasis.

***To prove** that two lines or two angles are equal:*

1. *Draw as accurately as possible a figure which the statement of the theorem shows to be necessary.*

2. *Select two triangles which appear to be congruent and which contain as parts the lines or angles to be proved equal.*

It will often be found necessary to draw lines not mentioned in the theorem in order to form the two triangles needed. See, for example, the proof of Theorems 14, 15, and 16.

3. *Prove the two triangles congruent by the use of one of the four following theorems: 1, 3, 8, and 13.*

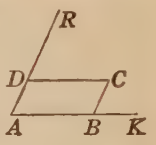
Later the student may use any one of six theorems, as Theorems 19 and 20 also relate to congruent triangles.

4. *State the conclusion: The required lines or angles are equal because they are corresponding parts of congruent triangles.*

The student should study carefully the application of the several steps of the preceding outline, especially in Theorems 14 and 15 and he should note their repeated application in

Theorems 16, 17, and 18. The list of exercises which follows § 96 will furnish material with which he can make a real beginning in applying the method for himself.

QUERY 1. In which of the theorems from 1 to 15 was the method described in § 89 first used? In which theorem was it used next?



QUERY 2. How many times has it been used in the theorems thus far?

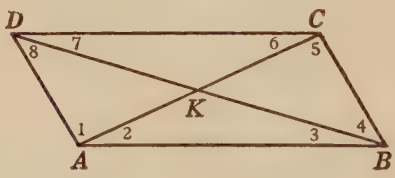
QUERY 3. If $ABCD$ is a parallelogram, does the point C lie within the angle KAR? Why?

QUERY 4. If the point C lies within the angle KAR, will the diagonal AC intersect the diagonal BD?

QUERY 5. Which of the first sixteen theorems were proved by superposition?

Theorem 17

90. *The diagonals of a parallelogram bisect each other.*



Given a parallelogram *ABCD* with its diagonals *AC* and *BD* intersecting at *K*.

To prove *that* $AK = KC$ *and* $DK = KB$.

HINTS. Show that a side and two adjacent $\measuredangle$ of $\triangle ABK$ are equal respectively to a side and two adjacent $\measuredangle$ of $\triangle CDK$.

Then use § 82 and § 27.

QUERY 1. In Theorems 1, 3, and 13 change the word *triangle* to *parallelogram* and then determine if the resulting statements are true.

QUERY 2. A diagonal is drawn in each of two quadrilaterals. If the two triangles of the first are respectively congruent to the two triangles of the second, are the quadrilaterals congruent? Explain.

Theorem 18 (*Converse of Theorem 17*)

91. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

Given the quadrilateral *ABCD* (using the figure of § 90) in which *AK* equals *KC* and *DK* equals *KB*.

To prove *that the quadrilateral ABCD is a parallelogram.*

HINTS. By the use of Theorem 1 show that $\triangle AKB$ is congruent to $\triangle DKC$. Then $\angle 6 = \angle 2$ by § 27, and AB is $\parallel$ to CD by § 64. In like manner, using $\triangle AKD$ and $\triangle BKC$, prove $AD \parallel$ to BC.

92. Drawing of figures. In drawing the figures necessary to prove any given theorem or exercise the student should make it a rule to observe the following points. A little experience will soon demonstrate their value.

1. *Draw the figures as accurately as possible.*

Later on methods of constructing many figures with theoretical exactness will be studied. Until then a sufficient degree of accuracy can be attained by the exercise of a little care and some judgment, and by the use of a ruler and a sharp-pointed pencil. A protractor may be used to lay off angles (see § 280).

Accuracy is especially desirable because a properly drawn figure often shows relations between its various lines and angles which are of great help in discovering a proof of the theorem. These relations are frequently obscured by a poorly drawn figure.

2. *Make the figures as general as possible.*

If the theorem refers to a triangle, draw one having the three sides unequal, for that is the most general triangle possible. If it calls for an *isosceles* triangle, do not draw one with *three* equal sides.

In the most general kind of quadrilateral no two sides are equal or parallel and no two angles are equal. Therefore, if the theorem specifies a quadrilateral, do not draw a parallelogram or a

quadrilateral having two sides parallel or equal. Similarly, if the theorem refers to a parallelogram, do not draw a rectangle, or a parallelogram all of whose sides are equal.

In drawing figures, then, avoid equal lines, parallel lines, equal angles, or right angles, unless the statement of the theorem says or implies that the figure should have them. And, furthermore, if a point is to be chosen on a line, do not select the mid-point of the line.

The student should form the habit of drawing general figures. Otherwise he will frequently be disappointed to find that what he thought was a valid proof is wholly incorrect, because it depends on an unwarranted assumption which crept into his reasoning without his noticing it. Very often he will find that he made the assumption because his figure was a special one.

93. Order in naming the vertices of a polygon. In stating exercises in which it is necessary to name a polygon of four or more sides by the letters at its vertices, it is customary to name the vertices in succession and in either order.

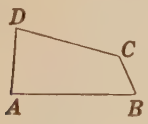


FIG. 1

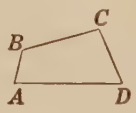


FIG. 2

FIG. 3

Thus the quadrilateral $ABCD$ is correctly represented by either Fig. 1 or Fig. 2 but not by Fig. 3. Unless this point is observed, the figure which results may appear to make the theorem wholly untrue or to give it a meaning other than the one intended.

94. Mutually equiangular and mutually equilateral polygons. It should also be noted that when two polygons are spoken of as being *mutually equiangular* the meaning is that the angles of one are equal respectively to the angles of the other. A like meaning holds for the phrase *mutually equilateral.*

95. Extending a line. In extending or producing a line it is understood that if we say line AB is produced to K, we mean that AB is extended beyond B to K so that B lies between A and K.

96. On methods of study. In some of the following exercises the method of § 89 is not needed. In others it is necessary, and if properly used will result in a solution. In still others the method of § 89 must be used first, followed by the application of theorems other than those on congruent triangles to complete the proof.

The necessity of application in the solution of the following list of exercises cannot be made too emphatic, for an hour's study daily at this point will go far to insure success and save many hours later. If the student finds that he can solve few or none of the exercises in the following list without help, it may be that he has not applied himself properly to the work which precedes, but has inferred that he could learn geometry by listening attentively to proofs worked out by others. Success in geometry is measured by the acquisition of power to solve original exercises. This will come only with sufficient attentive study.

Exercises on Congruent Triangles and Parallelograms

60. In the triangle ABC the line KR, parallel to AB, cuts AC in K and BC in R. Prove that the angles of the triangle ABC are equal respectively to the angles of the triangle KRC.

61. In the parallelogram $ABCD$ lines are drawn from B and D, the ends of the shorter diagonal, perpendicular to AC and meeting it in K and R respectively. Prove that the angles of the triangle BKC are equal respectively to the angles of the triangle DRA.

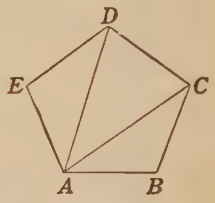
62. $ABCD$ is a parallelogram. AB and CD are produced in opposite directions the same distance to K and R respectively. Prove that the triangle ADR is congruent to the triangle CBK.

63. The line joining the mid-points of two opposite sides of a parallelogram divides it into two parallelograms.

64. AC is the longer diagonal of the parallelogram $ABCD$. From C and D perpendiculars are drawn to AB or AB produced, meeting it in K and R respectively. Prove that the triangles BKC and ADR are mutually equiangular.

65. $ABCD$ is a parallelogram and O the mid-point of the diagonal AC. A line is drawn through O cutting DC in K and AB in R. Prove that KO equals OR.

66. K is the mid-point of BC in the parallelogram $ABCD$. Line DK produced intersects AB produced in H. Prove that the triangle KCD is congruent to the triangle KBH.



67. $ABCDE$ is a pentagon which is equilateral and equiangular. Draw AC and AD and prove that the triangle ABC is congruent to the triangle AED and that the diagonals AC and AD are equal.

68. In the quadrilateral $ABCD$ the diagonals intersect at K so that KA equals KB and KC equals KD. Prove that BC equals AD.

69. In Ex. 61 prove that BK equals DR.

70. In Ex. 64 prove that CK equals DR.

71. The mid-point of one side of a parallelogram is joined to a vertex not adjacent to the side and the mid-point of the opposite side is joined to the vertex opposite the first. Prove that the two lines thus determined are equal and parallel.

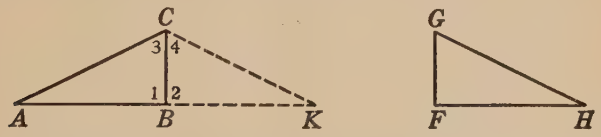
72. AC and BD, the diagonals of a parallelogram, meet in K. Points F, G, H, and L are the mid-points of AK, BK, CK, and DK respectively. Lines FG, GH, HL, and LF are drawn. Prove that $FGHL$ is a parallelogram.

73. If two opposite sides of a parallelogram are produced by the same distance in opposite directions and their ends joined to vertices so as to form a quadrilateral, it will be a parallelogram.

74. $ABCD$ is a parallelogram. R is taken on AB and K on CD so that BR equals DK. Prove that $AKCR$ is a parallelogram.

Theorem 19

97. *Two right triangles are congruent if the hypotenuse and another side of the first are equal respectively to the hypotenuse and another side of the second.*



Given the right triangles *ABC* and *FGH*, in which the hypotenuse *AC* equals the hypotenuse *HG* and the side *CB* equals the side *GF*.

To prove *that* $\triangle ABC$ *is congruent to* $\triangle FGH$.

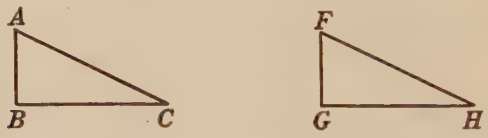
Proof

1. Place $\triangle FGH$ adjacent to $\triangle ABC$ in such a way that GF coincides with its equal CB, the point G falling on C and the point F on B. The point H will then fall at some point K.	1. § 20.
2. $\angle 1 = \angle 2 =$ a rt. $\angle$.	2. Hypothesis.
3. $\angle 1 + \angle 2 =$ a st. $\angle$.	3. § 40.
4. ABK is a straight line.	4. § 35.
5. $ABKC$ is a triangle.	5. § 16.
6. In $\triangle AKC$, $AC = CK$.	6. Hypothesis.
7. $\angle A = \angle K$.	7. § 29.
8. Hence $\triangle ABC$ is congruent to $\triangle BCK$.	8. §.50.
9. Therefore $\triangle ABC$ is congruent to $\triangle FGH$.	9. § 32.

QUERY. How could Theorem 19 have been proved if GH and FH had been given equal to CA and BA respectively?

Theorem 20

98. *Two right triangles are congruent if a side about the right angle and an acute angle of the first are equal respectively to a side about the right angle and a corresponding angle of the second.*



Case I. *When the acute angle is opposite the given side.*

Given the right triangles *ABC* and *FGH* with the side *BC* equal to the side *GH*, the angle *B* and the angle *G* right angles, and the acute angle *A* equal to the corresponding acute angle *F*.

To prove *that* $\triangle ABC$ *is congruent to* $\triangle FGH$.

Proof

1. In $\triangle ABC$ and FGH,	$\angle A = \angle F$,	1. Hypothesis.
	$\angle B = \angle G$.	
2.	$\angle C = \angle H$.	2. Why?
3.	$BC = GH$.	3. Hypothesis.
4. $\triangle ABC$ is congruent to $\triangle FGH$.		4. § 82.

Case II. *When the acute angle is adjacent to the given side.*

Given as in Case I except that the angle *C* equals the angle *H*.

HINT. Apply § 82.

QUERY. What is the precise meaning of the word *corresponding* in the statement of Theorem 20? Illustrate.

EXERCISE **75.** Give in full the details of Case II.

Exercises

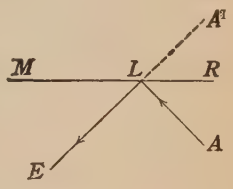
76. Two adjacent angles of a parallelogram are supplementary.

77. The lines joining the mid-points of adjacent sides of a parallelogram form a parallelogram.

78. The bisectors of two adjacent angles of a parallelogram intersect each other at right angles.

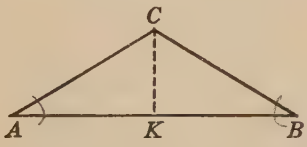
79. Two parallelograms are equal if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

80. The image of an object A seen in a plane mirror by the eye at E appears to be at A^1, as far behind the mirror MR as A is in front of it and so that AA^1 is perpendicular to MR. That is, rays of light from A strike the mirror at L and are reflected to E. Show that $\angle ALR = \angle ELM$.



Theorem 21 (*Converse of Theorem 2*)

99. *If two angles of a triangle are equal, the sides opposite those angles are equal.*



Given the triangle *ABC* with the angle *A* equal to the angle *B*.

To prove that $AC = BC$.

Hint. Let CK be $\perp$ to AB. Then use § 98.

Note. It is customary to call that side of an isosceles triangle which is not equal to either of the other two the *base*. And frequently that one of the three vertices of an isosceles triangle which lies opposite the base is called the *vertex*.

EXERCISES

81. In an isosceles triangle a line cutting two sides and parallel to the third side forms another isosceles triangle.

82. Prove that the exterior angles at the base of an isosceles triangle are equal.

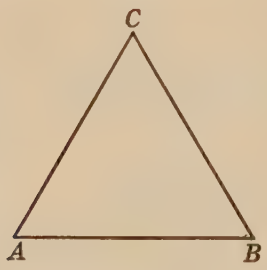
QUERY. If a quadrilateral has two equal angles, has it two equal sides?

100. Equilateral triangle. An *equilateral* triangle is a triangle which has three equal sides.

101. Equiangular triangle. An *equiangular* triangle is a triangle which has three equal angles.

Theorem 22

102. *If a triangle is equilateral, it is equiangular.*



Given the triangle *ABC* in which *AB* equals *BC* equals *CA*.

To prove that $\angle A = \angle B = \angle C$.

HINT. Apply Theorem 2 twice.

EXERCISE **83.** The sides AB, BC, and CA of the equilateral triangle ABC are extended by the same length to K, R, and L respectively. Prove that the triangle whose vertices are K, R, and L is equiangular.

Theorem 23 (*Converse of Theorem 22*)

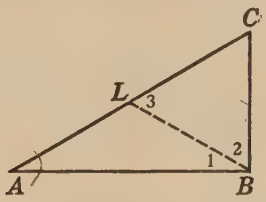
103. *If a triangle is equiangular, it is equilateral.*

HINT. Use the figure of § 102 and apply Theorem 21 twice.

104. *Corollary.* *Each angle of an equilateral or an equiangular triangle is 60°.*

Theorem 24

105. *If one acute angle of a right triangle is thirty degrees, the side opposite is half the hypotenuse.*



Given the right triangle *ABC* in which the angle *A* is 30° and *AC* is the hypotenuse.

To prove *that* $BC = \frac{1}{2}AC$.

Proof

1.	$30° + \angle C = 90°$.	1. § 75 and hypothesis.
2.	Hence $\angle C = 60°$.	2. § 51.
3.	Let BL be a line making $\angle LBA$, or $\angle 1$, $= 30°$.	3. $\angle ABC$ is greater than $\angle A$.
4.	$AL = BL$.	4. § 99.
5.	$\angle 2 = 60°$.	5. $\angle 1 = 30°$, and $\angle ABC = 90°$.
6.	$\angle 3 = 60°$.	6. § 69.
7.	$BL = LC = BC$.	7. From 2, 5, and 6, and § 103.
8.	$AL = LC = \frac{1}{2}AC$.	8. § 65.
9.	$BC = \frac{1}{2}AC$.	9. Statements 7 and 8.

QUERY. What is the converse of Theorem 24?

QUERY 1. If in Theorems 22 and 23 the word *polygon* replaces the word *triangle*, would the resulting statement be true?

QUERY 2. How many diagonals has a polygon of four sides? five sides? six sides? seven sides?

EXERCISES

84. Prove that a line cutting two sides of an equilateral triangle and parallel to the third side forms an equilateral triangle.

85. $ABCDEF$ is a regular hexagon. Prove that the lines AC, CE, and EA form an equilateral triangle.

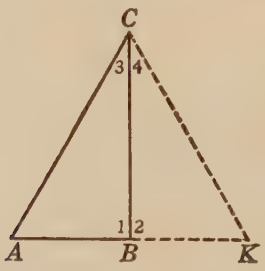
86. Prove that the longer diagonals of a regular hexagon bisect the angles at the vertices which they connect.

87. $ABCDEF$ is a regular hexagon. Prove that if the diagonals AD and BE are drawn, two equilateral triangles are formed.

88. In parallelogram $ABCD$ the angle B is 120°, and the diagonal BD makes a right angle with AD. If DK is perpendicular to AB at K, prove that BD is twice DK.

Theorem 25 (*Converse of Theorem 24*)

106. *If one side of a right triangle is half the hypotenuse, the angle opposite that side is thirty degrees.*



Given the right triangle *ABC*, in which the side *AB* is equal to one half the hypotenuse *AC*.

To prove *that* $\angle ACB = 30°$.

Proof

1. Extend AB to K, making $BK = AB$.	1. § 95.
2. Then draw CK.	2. § 19.
3. In $\triangle ABC$ and KBC, $\angle 1 = \angle 2$.	3. § 40.
4. $BC = BC$.	4. Why?
5. And $AB = BK$.	5. By construction.
6. $\triangle ABC$ is congruent to $\triangle KBC$.	6. § 25.
7. $AC = KC$.	7. § 27.
8. $AB = BK = \frac{1}{2} AC$.	8. Construction and hypothesis.
9. $(AB+BK)$ or $AK = AC = KC$.	9. Statements 7 and 8.
10. $\angle A = \angle K = \angle ACK = 60°$.	10. §§ 102 and 106.
11. $\angle 3 = \angle 4$.	11. § 27.
12. $\angle 3 = \frac{1}{2} \angle ACK = 30°$.	12. Statements 10 and 11.

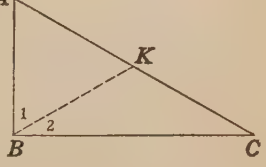
107. Trapezoid. A *trapezoid* is a quadrilateral two and only two of whose sides are parallel.

Exercises

89. AK, perpendicular to the base BC of the equilateral triangle ABC, meets BC in K; and KR, perpendicular to AB, meets AB in R. Prove that AK is twice KR.

90. $ABCD$ is a trapezoid with AB parallel to CD. If DK is perpendicular to AB at K and makes AK equal to one half AD, find the number of degrees in the angle ADC.

91. Using the adjacent figure and a different method from that given in the text, prove Theorem 25 as follows. Let BK be a line making $\angle 1 = \angle A$. $\angle A + \angle C = ?$ $\angle 1 + \angle 2 = ?$ What, then, is the relation between $\angle C$ and $\angle 2$? between AK and BK? etc.



108. Median. A *median* of a triangle is a line which joins any vertex to the mid-point of the opposite side.

NOTE. Among the more important elements of good writing or speaking are brevity, completeness, and a logical order of arrangement. The student's control of these three elements will be greatly increased if his attitude toward geometry is correct and if his part of the work involves the thinking out and the preparation of oral and written demonstrations of exercises. In this work care should be taken to avoid needless repetition, especially in the longer proofs. When in the course of a demonstration two lines or two angles can be proved equal just as two other lines or two other angles have been proved equal, it is best to omit all details in the second case and say, "In like manner it can be proved that $AB = KB$, or that $\angle a = \angle r$." On the other hand, important details should not be left out. The student should state clearly how any auxiliary lines are drawn and not assume that they are drawn in a certain way. All vital reasons should be quoted. Omission of a single one may ruin an otherwise perfect proof. Lastly the student should avoid a haphazard arrangement of the matter of a proof. A single statement not in its proper place may be fatal to a demonstration.

EXERCISES ON RIGHT, ISOSCELES, AND EQUILATERAL TRIANGLES

92. If a line bisects the vertical angle of an isosceles triangle, it bisects the base and is perpendicular to it.

93. The median from the vertex of an isosceles triangle is perpendicular to the base and bisects the vertical angle.

94. If a line from the vertex of an isosceles triangle is perpendicular to the base, it bisects the base and the vertical angle.

95. If any angle of an isosceles triangle is 60°, the triangle is equilateral.

96. The perpendiculars drawn from the middle point of the base to the equal sides of an isosceles triangle and terminating in them are equal.

97. The bisectors of the equal angles of an isosceles triangle intersect, and form, with its base, an isosceles triangle.

98. The medians drawn from the vertices of the equal angles of an isosceles triangle to the opposite sides are equal.

99. The perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides and terminating in them are equal.

100. State and prove the converse of the theorem of Ex. 99.

101. If a triangle is isosceles, the lines from the extremities of the base bisecting the equal angles and terminating in the opposite sides are equal.

102. State the converse of the theorem of Ex. 101.

NOTE. The student will observe that he is not asked to prove the theorem just stated. The theorem is true and its proof may appear to be easy. It is really very difficult to prove it, however, even by the indirect method of § 152, if one is limited to the theorems of Book I. A direct proof by Ex. 116 of the Supplementary Exercises on page 296 is not so difficult to obtain.

103. If the angles adjacent to the longer of the two parallel sides of a trapezoid are equal, the nonparallel sides are equal.

HINT. What two lines forming two right triangles might be of use? or what single line forming a parallelogram?

104. State the converse of the theorem of Ex. 103 and, if true, prove it.

105. One angle formed by the bisectors of two angles of an equilateral triangle is double the third angle of the triangle.

HINT. It is frequently helpful, as here, to determine the number of degrees in each angle of the figure and to write it plainly within the angle. Often the proof of the theorem will then be obvious.

106. ABC is an equilateral triangle. The bisectors of the angles A and C meet at K. The line KR is drawn parallel to AB, meeting AC at R, and KL is drawn parallel to BC, meeting AC at L. Prove that AR, RK, RL, KL, and CL are equal. (See the hint to Ex. 105.)

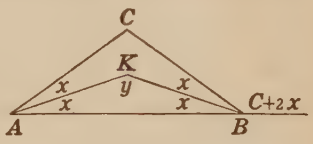
107. The line through the vertex of an isosceles triangle parallel to the base bisects the exterior angles at the vertex.

108. Has the theorem of Ex. 107 a converse? more than one? If so, state and prove them.

109. Under what conditions may a theorem have more than one converse?

110. ABC is any triangle. AKB and ARC are equilateral triangles adjacent to the triangle ABC and having the sides AB and AC respectively in common with it. Show that CK equals BR.

111. The obtuse angle between the bisectors of the equal angles of an isosceles triangle is equal to an exterior angle at the base of the triangle.



Hints. This involves a geometrical identity and is best attacked algebraically. Why can the angles be marked as in the figure? Does $y + 2x = \angle C + 2x + 2x$? Why? Conclusion?

Try to apply the method here illustrated wherever possible.

112. If each of the angles at the base of an isosceles triangle is one fourth the vertical angle, every line perpendicular to the base which does not pass through the vertex forms an equilateral triangle with the other two sides, one being produced. (See hint to Ex. 105.)

113. K is any point on the base BC of the isosceles triangle ABC. The side AC is produced from C to R so that CR equals CK, and KR is drawn meeting AB at L. Prove that the angle ALR equals three times the angle ARL.

Hints. Let a be the number of degrees in the angle R. What other two angles can also be marked a degrees? What two angles can be marked $2a$ degrees? What angle $3a$ degrees?

114. In the triangle ABC the bisectors of the angle A and the exterior angle at B intersect at K. Prove that the angle AKB is one half of the angle C.

109. Square. A *square* is a rectangle whose sides are equal.

110. Rhombus. A *rhombus* is a parallelogram whose sides are equal and whose angles are not right angles.

Theorem 26

111. *The diagonals of a rectangle are equal.*

Given the rectangle *ABCD* with the diagonals *AC* and *BD*.

To prove that $AC = BD$.

HINT. Select two triangles, one having AC as a side and one having BD as a side. Prove that these ▲ are congruent by § 25. Conclusion?

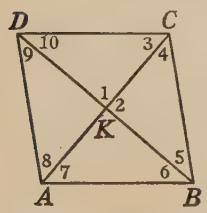
112. *Corollary.* *The diagonals of a square are equal.*

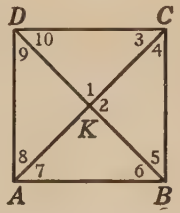
Theorem 27

113. *The diagonals of a rhombus or of a square*

(1) *meet at right angles;*

(2) *bisect the four angles of each.*





Given the rhombus *ABCD* and the square *ABCD* with diagonals *AC* and *BD* intersecting at *K*.

To prove that (1) $\angle 1 = \angle 2 = a\ rt.\angle$.

(2) $\angle 3 = \angle 4$, $\angle 5 = \angle 6$, $\angle 7 = \angle 8$, $\angle 9 = \angle 10$.

HINTS. (1) Which △ contains $\angle 1$? Which △ contains $\angle 2$? Prove these ▲ congruent. Then is $\angle 1$ equal to $\angle 2$? Why? Conclusion?

114. *Corollary.* *The diagonals of a square make an angle of 45° with each of the four sides.*

QUERY 1. Do the diagonals of a parallelogram meet at right angles? Do they bisect the angles of the parallelogram? Are they equal?

QUERY 2. If the diagonals of a quadrilateral are equal, is the figure necessarily a parallelogram?

EXERCISES

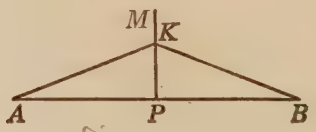
115. If the diagonals of a quadrilateral are equal and bisect each other at right angles, the quadrilateral is a square.

116. If the diagonals of a quadrilateral bisect each other at right angles and the shorter diagonal equals one side, the quadrilateral is a rhombus having one angle 120°.

115. Mid-perpendicular. If a line is perpendicular to another line at its middle point, the first line is called the *mid-perpendicular* of the second.

Theorem 28

116. *If a point is on the mid-perpendicular of a line, it is equidistant from the ends of the line.*



Given *MP* the perpendicular erected at the mid-point of *AB*, *K* any point on *MP*, and the lines *KA* and *KB*.

To prove that $KA = KB$.

HINT. Prove $\triangle APK$ congruent to $\triangle BPK$.

EXERCISE **117.** ABC and ABK are triangles on opposite sides of the common base AB. If line AB is the mid-perpendicular of KC, prove that the triangle ABC is congruent to the triangle ABK.

Theorem 29 *(Converse of Theorem 28)*

117. *If a point is equidistant from the ends of a line, it is on the mid-perpendicular of the line.*

Given the line *KR* and the point *A* so that *AK* equals *AR*.

To prove *that A lies on the mid-perpendicular of KR.*

Proof

1. Let AL be $\perp$ to KR.	1. § 42.
2. Then in $\triangle AKL$ and ARL, $AL = AL$.	2. Why?
3. And $AK = AR$.	3. Why?
4. $\triangle AKL$ is congruent to $\triangle ARL$.	4. Why?
5. $KL = LR$.	5. Why?
6. AL is the mid-perpendicular of KR.	6. § 115.

118. ***Corollary.*** *Two points each equally distant from the extremities of a line determine the mid-perpendicular of the line.*

HINT. Apply § 117 twice, followed by § 41.

QUERY 1. Is the point of intersection of the mid-perpendiculars to two sides of a triangle equally distant from the three vertices?

QUERY 2. If a point is equally distant from all three vertices of a triangle, on how many of the mid-perpendiculars to the sides does it lie?

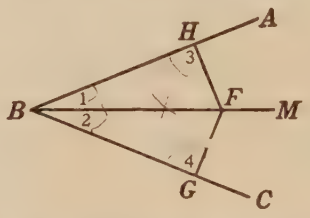
119. Concurrent lines. Three or more lines which have one point in common are said to be *concurrent.*

EXERCISE **118.** Prove that the mid-perpendiculars to the three sides of a triangle are concurrent.

120. Distance to a line. The *distance* from a point to a line is the length of the *perpendicular* from the point to the line.

Theorem 30

121. *If a point is on the bisector of an angle, it is equally distant from the sides of the angle.*



Given the angle *ABC* with *BM* its bisector, *F* a point on *BM*, and *FH* and *FG* the distances from *F* to *AB* and *CB* respectively.

To prove that $FH = FG$.

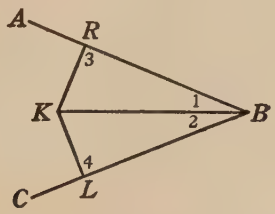
Proof

1. In $\triangle BFH$ and BFG, $\angle 1 = \angle 2$.	1. Why ?
2. $\angle 3$ and $\angle 4$ are rt. $\angle$s.	2. § 120.
3. $BF = BF$.	3. Why ?
4. $\triangle BFH$ is congruent to $\triangle BFG$.	4. § 50.
5. $FH = FG$.	5. Why ?

EXERCISE **119.** CD is the shorter base of the trapezoid $ABCD$. The bisectors of the angle C and the angle D intersect at K. Prove that K is the same distance from BC, CD, and DA.

Theorem 31 *(Converse of Theorem 30)*

122. *If a point is equally distant from the sides of an angle, it is on the bisector of the angle.*



Given the angle *ABC* and the point *K* such that the distances *KR* and *KL* are equal.

To prove *that the point K is on the bisector of* $\angle ABC$.

Proof

1. Draw BK.	1. § 19.
2. Then in $\triangle BKR$ and BKL, $BK = BK$.	2. Why?
3. $KR = KL$.	3. Why?
4. $\angle 3 = \angle 4 = 90°$.	4. § 120.
5. $\triangle BKR$ is congruent to $\triangle BKL$.	5. § 97.
6. $\angle 1 = \angle 2$.	6. Why?
7. KB bisects $\angle ABC$.	7. § 28.

QUERY 1. Is the point of intersection of the bisectors of two angles of a triangle equidistant from the sides of the triangle?

QUERY 2. If a point is equidistant from the three sides of a triangle, on the bisectors of what angles does it lie?

EXERCISE **120.** Prove that the bisectors of the three angles of a triangle are concurrent.

QUERY 3. Are the bisectors of the angles of a square concurrent?

QUERY 4. Are the bisectors of the angles of a parallelogram concurrent?

123. Convex and reëntrant polygons. A polygon is called *convex* if each of its angles is less than two right angles, and *reëntrant* if any one of its angles is greater than two right angles.

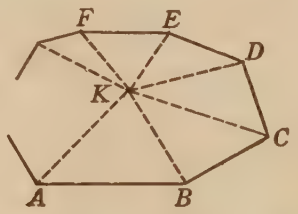
QUERY 1. Does the following statement correctly define a convex polygon? If a straight line can cut the boundary of a polygon in but two points, the polygon is convex.

QUERY 2. How can a reëntrant polygon be defined without mentioning its angles?

124. ***Axiom VI.*** *If equals are multiplied by equals, the results are equal.*

Theorem 32

125. *If n is the number of sides of a convex polygon and s is the sum of its interior angles, then $s = (2n - 4)$ right angles.*



Given any convex polygon *ABCDEF* · · · having *n* sides.

To prove *that the sum of the interior angles of ABCDEF · · · is $(2n - 4)$ rt. ∡s.*

Proof

1. From K, any point within the polygon, draw a line to each vertex forming n triangles.	1. § 19.
2. The sum of the angles of each triangle $= 2$ rt. ∡s.	2. § 66.
3. The sum of the angles of the n triangles $= 2n$ rt. ∡s.	3. § 124.

4. Angles about the point K $= 4$ rt. $\measuredangle$.	4. § 40.
5. Interior angles of polygon $= s = 2n$ rt. $\measuredangle - 4$ rt. $\measuredangle$.	5. § 51. These angles at K are included in the $2n$ rt. $\measuredangle$, but they are not included in the angles of the polygon.

Exercise **121.** Draw a reëntrant polygon and determine whether the method of proving Theorem 32 can be used to obtain a demonstration of the theorem for such a polygon. Is Theorem 32 true for a reëntrant polygon?

126. Equiangular polygon. A polygon whose interior angles are all equal is an *equiangular* polygon.

Query. If a polygon has n vertices, how many sides has it? Conversely?

Exercises

122. If one interior angle of an equiangular polygon of n sides is x rt. $\measuredangle$, prove that $x = \dfrac{2n-4}{n}$.

123. A polygon has 18 sides. Find in degrees the sum of its interior angles.

124. The sum of the interior angles of a polygon is 36 right angles. How many sides has it?

Hint. $36 = 2n - 4$, etc.

125. Find the number of sides of a polygon the sum of whose interior angles is 2520°.

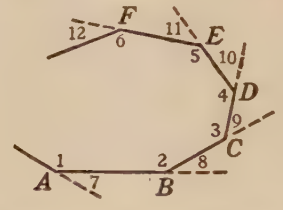
126. A regular polygon has 10 sides. How many degrees in each interior angle?

127. One interior angle of a regular polygon contains 168°. How many sides has the polygon?

Hint. $\dfrac{168}{90} = \dfrac{2n-4}{n}$, etc.

Theorem 33

127. *The sum, S, of the exterior angles of any convex polygon, counting one at each vertex, is four right angles.*



Given the convex polygon *ABCDE* · · · having *n* sides.

To prove *that the sum of the exterior angles, one at each vertex, is 4 rt. ∡s.*

Proof

1. Extend the sides, forming one exterior angle at each vertex.	1. § 68.
2. The sum of the exterior angle and the interior angle at each vertex is 2 rt. ∡s. For example, $\angle 1 + \angle 7 = 2$ rt. ∡s.	2. § 40.
3. Therefore the sum of the n interior angles and the n exterior angles is $2n$ rt. ∡s.	3. Why?
4. But the sum of the interior angles alone is $(2n - 4)$ rt. ∡s.	4. § 125.
5. Therefore the sum, S, of the exterior angles of any polygon, expressed in right angles, is $S = 2n - (2n - 4) = 4$.	5. § 51.

Exercises

128. Is Theorem 33 true for reëntrant polygons? Explain.

129. How many degrees are there in each exterior angle of a regular octagon? a regular decagon?

130. Find the number of sides of a polygon if the sum of its exterior angles equals the sum of its interior angles.

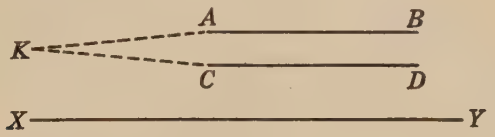
HINTS. Let n be the number of sides of the polygon. Then, by §§ 125 and 127, $4 = 2n - 4$.

131. Find the number of sides of a polygon if the sum of its exterior angles is twice the sum of its interior angles.

132. Find the number of sides of a polygon if the sum of its interior angles is twice the sum of its exterior angles.

Theorem 34

128. *If two lines are parallel to a third line, the two lines are parallel to each other.*



Given the lines *AB*, *CD*, and *XY* in the same plane, with *AB* and *CD* parallel to *XY*.

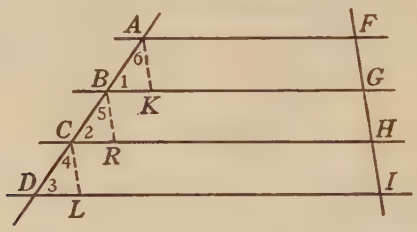
To prove that *AB is* ∥ to *CD*.

Proof

1. The lines AB and CD either meet or do not meet.	1. No other possibility exists.
2. If they meet, as at K, there will then be two lines through K ∥ to XY.	2. Hypothesis.
3. But this is impossible.	3. § 45. Postulate V. Through a given point outside a line, one line parallel to it exists, and only one.
4. AB and CD cannot meet.	4. Remaining alternative from statement 1.
5. AB is ∥ to CD.	5. § 43.

Theorem 35

129. *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on any other transversal.*



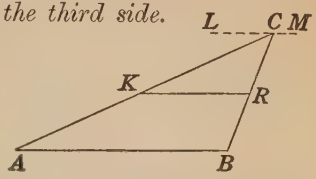
Given the parallels *AF*, *BG*, *CH*, and *DI* intercepting the equal segments *AB*, *BC*, and *CD* on the transversal *AD* and intercepting the segments *FG*, *GH*, and *HI* on another transversal.

To prove *that* $FG = GH = HI$.

Proof

1. Through A, B, and C draw $\parallel$s to FI, meeting BG, CH, and DI in K, R, and L respectively.	1. § 45.
2. Then AK, BR, and CL are $\parallel$ lines.	2. § 128.
3. In $\triangle ABK$, BCR, and CDL, $\angle 6 = \angle 5 = \angle 4$.	3. § 59.
4. Also $\angle 1 = \angle 2 = \angle 3$.	4. § 59.
5. $AB = BC = CD$.	5. Hypothesis.
6. Therefore $\triangle ABK$, BCR, and CDL are congruent.	6. Why?
7. $AK = BR = CL$.	7. Why?
8. But $AK = FG$, $BR = GH$, and $CL = HI$.	8. § 85.
9. Therefore $FG = GH = HI$.	9. § 32.

130. ***Corollary.*** *If a line bisects one side of a triangle and is parallel to another side, it bisects the third side.*

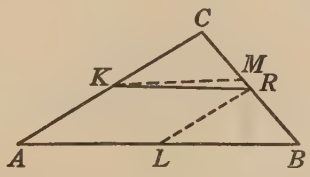


HINTS. Let K be the mid-point of AC and let KR be $\parallel$ to AB. Draw LM through C parallel to AB. Then LM, KR, and AB are $\parallel$ lines. Why? Therefore $CR = RB$. Why?

QUERY. If a line is parallel to one of two parallels, it is parallel to the other. Contrast this statement with the statement of Theorem 34.

Theorem 36

131. *The line which joins the mid-points of two sides of a triangle is parallel to the third side and equal to one half of it.*



Given the triangle *ABC* in which the line *KR* joins the mid-points of *AC* and *BC*.

To prove *that* KR *is* $\parallel$ *to* AB *and* $= \frac{1}{2} AB$.

Proof

1. Draw $KM \parallel$ to AB.	1. § 45.
2. M is the mid-point of CB.	2. § 130.
3. KM and KR coincide.	3. Why?
4. KR is $\parallel$ to AB.	4. Why?
5. Draw $RL \parallel$ to CA.	5. § 45.
6. L is the mid-point of AB, and $AL = BL = \frac{1}{2} AB$.	6. § 130.
7. $ALRK$ is a parallelogram.	7. Why?
8. $KR = AL$.	8. § 85.
9. $KR = \frac{1}{2} AB$.	9. § 65.

Exercises

133. Find the number of sides of a polygon if the sum of its interior angles exceeds the sum of its exterior angles by 34 right angles.

134. Find the number of sides of a polygon if the sum of its interior angles exceeds the sum of its exterior angles by 1260°.

135. Find the number of sides of a polygon if it is regular and one exterior angle is one seventeenth of one interior angle.

136. $ABCD$ is a parallelogram. AD and BC are divided into five equal parts, and the corresponding points of division are joined. Show that these lines divide the diagonals of the parallelogram into five equal parts.

137. If K is any point on AB of the triangle ABC and R is the mid-point of AK, L the mid-point of AC, and M the mid-point of CK, then $KRLM$ is a parallelogram.

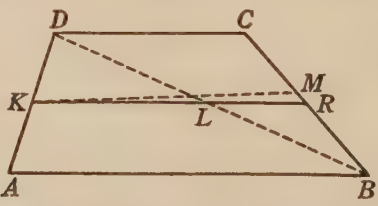
138. The lines joining the mid-points of the sides of a triangle divide it into four congruent triangles.

139. The line bisecting two sides of a triangle bisects all lines drawn to the third side from the opposite vertex.

132. Bases of a trapezoid. The *bases* of a *trapezoid* are the two parallel sides.

Theorem 37

133. *The line joining the mid-points of the nonparallel sides of a trapezoid is parallel to the bases and equal to half their sum.*



Given the trapezoid *ABCD* with *KR* joining the mid-points of the nonparallel sides *AD* and *CB*.

To prove that KR *is* $\parallel$ *to* AB *and* DC *and* $= \frac{1}{2}(AB + DC)$.

Proof

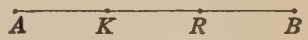
1. Draw $KM \parallel$ to AB.	1. § 45.
2. $CM = MB$.	2. § 129.
3. M and R coincide.	3. Why?
4. KR coincides with KM.	4. Why?
5. KR is $\parallel$ to AB.	5. Why?
6. KR is $\parallel$ to DC.	6. Why?
7. Draw BD, cutting KR at L.	7. § 19.
8. L is the mid-point of BD.	8. § 130.
9. $KL = \frac{1}{2} AB$,	9. § 131.
10. $LR = \frac{1}{2} DC$.	10. Why?
11. $KL + LR = \frac{1}{2} AB + \frac{1}{2} DC$.	11. Why?
12. $KR = \frac{1}{2}(AB + DC)$.	12. Why?

Exercises

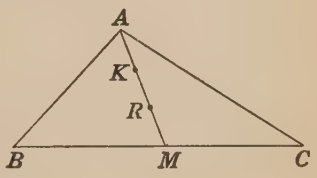
140. If a line bisects one nonparallel side of a trapezoid and is parallel to one base, it bisects the other nonparallel side also.

141. The line joining the mid-points of two adjacent sides of a quadrilateral is equal to one half of one of the diagonals of the quadrilateral.

134. Trisection. If a magnitude is divided into three equal parts, it is said to be *trisected*.



If $AK = KR = RB$, AB is trisected, K and R being the points of trisection.



Let AM be the median of the triangle ABC, with K and R the points of trisection. Then R, the point nearest the side to which the median is drawn, is called the *basal point of trisection*, and K is called the *vertical point of trisection*.

135. Inequalities. In a number of theorems now to be proved, *inequalities* occur. To the handling of inequalities, either alone or in connection with equations, certain laws of operation (axioms) apply.

The signs of inequality are $>$, which is read "is greater than," and $<$, which is read "is less than."

136. *Axiom VII.* *The whole is greater than any of its parts.*

137. *Axiom VIII.* *If the first of three magnitudes is greater than the second and the second is greater than the third, the first is greater than the third.*

In algebraic symbols this axiom states that if $a > b$ and $b > c$, then $a > c$.

138. Order of inequalities. Two inequalities are said to exist in the *same order* if the left member in each is the greater member or the less member, as the case may be.

Thus $9 > 7$ and $6 > 3$ exist in the same order; also $x < m$ and $c < a$ exist in the same order.

Two inequalities are said to be in the *reverse order* if the left member of one is its greater member and the right member of the other is its greater member, or vice versa.

Thus the inequalities $7 > 3$ and $8 < 10$ are in reverse order.

139. *Axiom IX.* *If the same number, positive or negative, is added to or subtracted from each member of an inequality, the results are unequal in the same order.*

140. *Axiom X.* *If both members of an inequality are multiplied or divided by the same positive number, the results are unequal in the same order.*

If $a > b$ and x is positive, Axioms IX and X may be stated in algebraic symbols thus: $a + x > b + x$, $a - x > b - x$, $ax > bx$, $\frac{a}{x} > \frac{b}{x}$.

141. ***Axiom XI.*** *If the corresponding members of two or more inequalities which are in the same order are added, the sums are unequal in the same order.*

Thus, if $a > x$ and $b > y$, then $a + b > x + y$.

142. ***Axiom XII.*** *If unequals are subtracted from equals, the results are unequal in the reverse order.*

Thus, if $a > b$, then $x - a < x - b$, while if $c < d$, $x - c > x - d$.

Exercise **142.** Write down one or more numerical inequalities (such as $8 > 3$) and test the truth of Axioms VIII to XII.

Query 1. If the members of an equation are subtracted from the corresponding members of an inequality, what is the form of the result? In what order is it?

Query 2. If the members of an inequality are subtracted from the corresponding members of an equation, what is the result? In what order is it?

Query 3. In the inequality $a + x > b$, may k be substituted for x if $k = x$?

Query 4. In an inequality may any term be replaced by its equal without making any other change? Illustrate.

Query 5. If a number is added to the greater member of an inequality, what is the result? If added to the less member?

Query 6. Change *added to* to *subtracted from* in Query 5 and answer.

Query 7. If in Axiom XII *subtracted from* is changed to *added to*, what other change must be made to make the resulting statement true?

Query 8. May a term be transposed from one member of an inequality to the other without destroying the inequality?

Query 9. May all the terms in each member of an equation be transposed to the opposite side of the sign of equality?

Query 10. In $7 - 3 + 8 > 4 - 2 + 1$ transpose every term and inspect the result.

Query 11. What conclusion may be drawn from the result in Query 10?

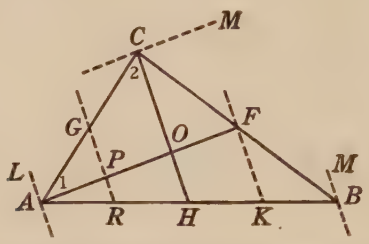
Query 12. If $x + 8 > 9$, show that $x > 1$.

Query 13. If $2x - 6 > 4$, show that $x > 5$.

Query 14. If $\frac{x}{2} + 5 < 6$, show that $x < 2$.

Theorem 38

143. *The medians of a triangle are concurrent in the basal point of trisection of each.*



Given the triangle *ABC* with the medians *AF* and *CH*, and *G* the mid-point of *AC*.

To prove *that*

(I) *AF and CH intersect at some point O.*

(II) $AF = 3\,OF$ *and* $CH = 3\,OH$.

(III) *The third median BG passes through O, making* $BG = 3\,OG$.

Proof

(I)

1. $\angle BAC + \angle BCA < 2$ rt. $\measuredangle$.	1. § 66.
2. $\angle 1 < \angle BAC$ and $\angle 2 < \angle BCA$.	2. § 136.
3. $\angle 1 + \angle 2 < \angle BAC + \angle BCA$.	3. § 141.
4. $\angle 1 + \angle 2 < 2$ rt. $\measuredangle$.	4. § 137.
5. *AF* and *CH* are not ∥ and must meet at some point *O*.	

(II)

1. Draw *LA*, *GR*, *FK*, and *MB* ∥ to *CH*, points *R* and *K* being on *AB*.	1. § 45.
2. Since $AG = GC$, $AR = RH$. In like manner, $BK = KH$.	2. § 130.

3.	$AH = HB.$	3. Hypothesis.
4.	$AR = RH = HK = KB.$	4. § 39.
5.	As $AR = RH = HK$, $AP = PO = OF.$ Then $AF = 3\,OF.$	5. § 129.
6.	In like manner it can be proved that $CH = 3\,OH.$	6. Why?

(III)

This proof shows that O is the basal point of trisection of CH and AF. In the same way BG and AF can be proved to intersect in their basal point of trisection. But each median of a triangle, as AF, has only one basal point of trisection. Therefore the three medians of a triangle intersect in that point.

Exercises on Medians and Mid-points

143. $ABCD$ is a parallelogram. The diagonals intersect in R. The line DK bisects AB at K and cuts AC in L. Prove that DL is twice LK and LR equals one sixth of AC.

144. If K is the middle point of the side BC of the triangle ABC and BR and CL are perpendiculars from B and C respectively to AK, produced if necessary, prove that BR equals CL.

145. If a median of a triangle is perpendicular to one side, the triangle is isosceles.

146. If two medians of a triangle are equal, the triangle is isosceles.

147. The side BC of the triangle ABC is produced to the point K. The angle ACB is bisected by the line CR, which meets AB at R. A line is drawn through R parallel to BC, meeting AC at L and the line bisecting the exterior angle ACK at G. Show that RL equals LG.

148. In a right triangle the median from the vertex of the right angle is one half the hypotenuse.

149. State the converse of the theorem of the preceding exercise and, if it is true, prove it.

150. BC is the base of the isosceles triangle ABC. The side BA is produced its own length to K, and KC is drawn. Prove that the triangle BCK is a right triangle.

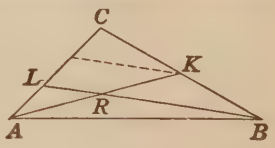
HINT. Note the equal angles in the figure and then consider the sum of the angles of the triangle BCK.

151. Draw KR from K on AC of triangle ABC to R on BC so that AK is one fourth of AC and BR is one fourth of BC. Prove that KR is three fourths of AB.

152. In the triangle ABC the line KR parallel to AB joins K on AC to R on BC and is equal to two fifths of AB. Prove AK equal to three fifths of AC.

153. If lines be drawn from any vertex of a parallelogram to the mid-points of the opposite sides, they will divide the diagonal which they intersect into three equal parts.

154. ABC is a triangle. K is the mid-point of BC and R the mid-point of AK. If BR produced meets AC in L, prove AL equal to one third of AC.

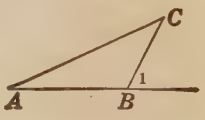


155. The lines joining the mid-points of the adjacent sides of a quadrilateral form a parallelogram.

156. The lines joining the mid-points of the opposite sides of a quadrilateral bisect each other.

157. K is taken on AB of the parallelogram $ABCD$ so that AK is one fifth of AB. The lines DK and AC intersect at R. Prove that AR is one sixth of AC.

144. ***Corollary to*** **§ 69.** *An exterior angle of a triangle is greater than either opposite interior angle.*

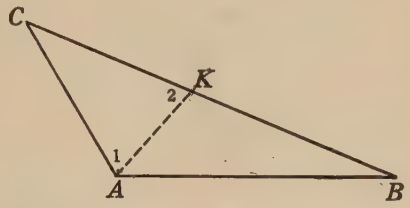


HINTS. $\angle 1 = \angle A + \angle C$. Therefore $\angle 1 > \angle A$ or $\angle C$, by Axiom VII, § 136.

EXERCISE **158.** In the triangle ABC the side AC equals the side BC. AK is drawn to any point K on BC. Prove that the angle AKC is greater than the angle CAB.

Theorem 39

145. *If two sides of a triangle are unequal, the angles opposite them are unequal and the greater angle lies opposite the greater side.*



Given the triangle *ABC* with *CB* greater than *CA*.

To prove *that* $\angle CAB > \angle B$.

Proof

1. On CB lay off CK equal to CA and draw AK.		1. § 19.
2. K lies between C and B.		2. CB is given greater than CA.
3.	$\angle CAB > 1$.	3. § 136.
4.	$CK = CA$.	4. Construction.
5.	$\angle 1 = \angle 2$.	5. § 29.
6.	$\angle 2 > \angle B$.	6. § 144.
7.	$\angle 1 > \angle B$.	7. § 65.
8. Therefore $\angle CAB > \angle B$.		8. § 137.

Exercises

159. In the triangle ABC the side AC equals BC. AK is drawn to any point K on BC. Prove by use of § 145 that the angle CKA is greater than the angle KAC.

160. If the diagonals of a quadrilateral are unequal and bisect each other at right angles, the figure is a rhombus.

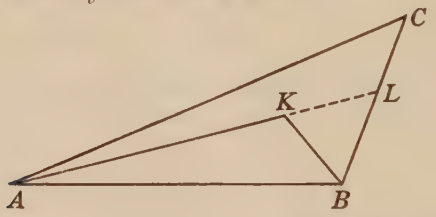
Hints. Let $ABCD$ be the quadrilateral, K the point of intersection of the diagonals, and AC greater than BD. Prove that $ABCD$ is a parallelogram with equal sides. Then, in the triangle ABK, show that the angle KBA is greater than the angle KAB.

146. ***Postulate VI.*** *Any side of a triangle is less than the sum of the other two sides.*

A still broader and more usual statement is: *A straight line is the shortest line between two points.*

Theorem 40

147. *The sum of any two sides of a triangle is greater than the sum of two lines drawn from a point within it to the extremities of the third side.*



Given the triangle *ABC* with the lines *KA* and *KB* drawn from the point *K* within the triangle to the extremities of *AB*.

To prove *that* $AC + CB > KA + KB$.

Proof

Extend AK to meet CB in some point L

1. $AC + CL > AK + KL$.	(1)	1. § 146.
2. Also $KL + LB > KB$.	(2)	2. § 146.
3. $AC + CL + LB + KL > AK + KL + KB$.		3. (1) + (2), § 141.
4. $AC + CL + LB > AK + KB$.	(3)	4. § 139.
5. $CL + LB = CB$.	(4)	5. Why?
6. $AC + CB > AK + KB$.		6. From (4) and (3), by § 65.

EXERCISES

161. In a quadrilateral any side is less than the sum of the other three sides.

162. $ABCDE$ is a pentagon. Draw CE and prove that the perimeter of $ABCE$ is less than that of the pentagon.

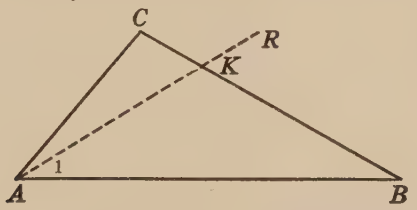
163. If the lines AB and CD intersect, then the sum of AB and CD is greater than the sum of AC and DB.

164. If from a point within a triangle lines are drawn to the vertices, their sum is greater than half the sum of the sides of the triangle.

HINT. Study the figure to obtain three inequalities which, if their corresponding members be added, will give a suggestive result.

Theorem 41

148. *If two angles of a triangle are unequal, the sides opposite them are unequal and the greater side lies opposite the greater angle.*



Given the triangle *ABC* in which the angle *CAB* is greater than the angle *B*.

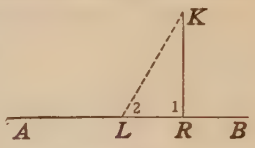
To prove *that* $BC > AC$.

Proof

Suppose AR to be the line through A such that $\angle RAB$ (or $\angle 1$) $= \angle B$.

1. $\angle A > \angle B$.	1. Hypothesis.
2. $\angle A > \angle 1$.	2. § 65.
3. Therefore AR lies between AB and AC and must intersect BC in some point, K.	3. § 136.
4. Now in $\triangle ABK$, $AK = BK$.	4. § 99.
5. But $AK + KC > AC$.	5. § 146.
6. $BK + KC\,(=BC) > AC$.	6. § 65.

149. *Corollary.* *The perpendicular from a point outside a straight line is the shortest line from the point to the line.*



HINTS. Let AB be any line, K any point outside AB, KR a $\perp$ to AB, and KL any other line from K to AB. Compare $\angle 1$ with $\angle 2$. Conclusion?

QUERY. How does the hypotenuse of a right triangle compare with each of the other two sides? Explain.

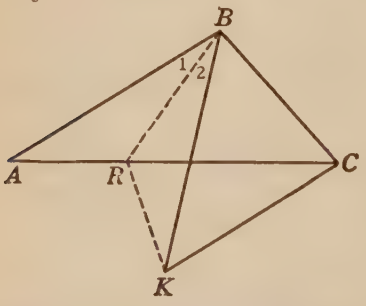
EXERCISES

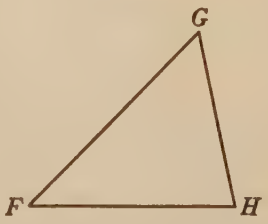
165. In the triangle ABC the side AC equals the side AB. If AK is any line from A to BC, prove that AK is less than AB.

166. A rectangle and a parallelogram have the same base and equal altitudes. Which has the greater perimeter? Prove it.

Theorem 42

150. *If two triangles have two sides of one equal respectively to two sides of the other and the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.*





Given the triangles *ABC* and *FGH*, in which *AB* equals *FG*, *BC* equals *GH*, and the angle *ABC* is greater than the angle *G*.

To prove that $AC > FH$.

Proof

1. Superpose $\triangle FGH$ upon $\triangle ABC$ so that the side GH will fall on its equal BC, the point G falling at B, the point H at C, and the point F at some point K.	1. § 20.
2. Line GF will fall within the angle ABC in the position BK.	2. By hypothesis, $\angle G < \angle ABC$.
3. Let the line BR bisect $\angle ABK$ and intersect AC at R.	3. § 28.
4. Draw RK.	4. § 19, Postulate I.
5. In $\triangle ABR$ and KBR, $BR = BR$.	5. Why?
6. $\angle 1 = \angle 2$.	6. Construction.
7. $AB = GF$, or its equal BK.	7. Hypothesis.
8. $\triangle ABR$ and $\triangle KBR$ are congruent.	8. § 25.
9. $RA = RK$.	9. § 27.
10. $CR + RK > CK$.	10. § 146.
11. $CR + RA > CK$.	11. § 65.
12. $AC > FH$.	12. Why?

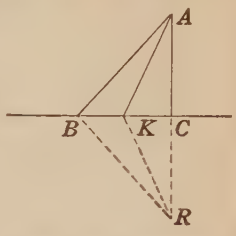
Exercises

167. If a parallelogram is not a rectangle, its diagonals are unequal.

168. Referring to the figure of Ex. 80, p. 51, suppose that the ray of light from A to the mirror was reflected to E from some other point than L. Show that the distance the light travels would be longer than ALE.

169. Two towns situated certain distances from a straight river bank each wish to secure a water supply from a common pumping station on the bank of the river, from which a pipe line is to run to each town. Where should the station be placed so that the pipe lines may be of the shortest possible length?

170. If two oblique lines drawn from a point to a straight line meet the line at unequal distances from the foot of the perpendicular drawn from the point to the line, they are unequal, and the more remote is the greater.



HINTS. Extend AC to R, making $CR = AC$, and draw RK and RB. Then $AB + RB > AK + RK$. Why? Conclusion?

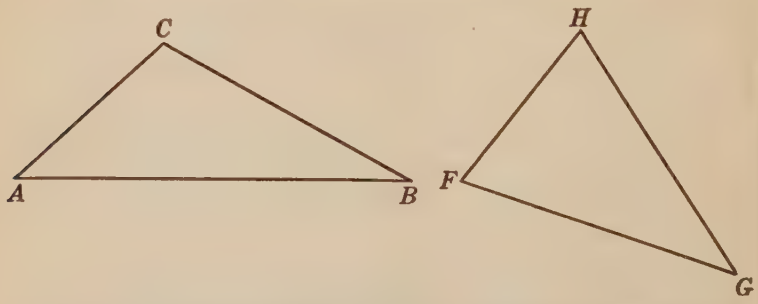
171. If from a point within a triangle lines be drawn to the vertices, their sum is less than the sum of the three sides of the triangle.

HINT. Use Theorem 40 and Axiom XI; then use Axiom X.

172. In the triangle ABC the median AK is drawn forming an acute angle AKB. Which is the greater, AB or AC?

Theorem 43

151. *If two triangles have two sides of one equal respectively to two sides of the other and the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.*



Given the triangles *ABC* and *FGH*, in which *AC* equals *FH*, *BC* equals *GH*, and *AB* is greater than *FG*.

To prove that $\angle C > \angle H$.

Proof

Let us suppose $\angle C < \angle H$; (1) $\angle C = \angle H$; (2) or $\angle C > \angle H$. (3)	No other possibilities exist.
1. If we suppose (1) is true, then it follows that $AB < FG$.	1. § 150.
2. But $AB > FG$.	2. Given.
3. Hence (1) is not true.	3. A *supposition* which by correct reasoning leads to a false conclusion is a false supposition.
4. If we suppose (2) is true, it follows that $\triangle ABC$ is congruent to $\triangle FGH$.	4. § 25
5. Then $AB = FG$.	5. Why?
6. But $AB > FG$.	6. Given.
7. Hence (2) is not true.	7. A *supposition* is false which makes it possible to prove that $AB = FG$.
8. Now one of the statements (1), (2), or (3) must be true.	8. There are no further alternatives.
9. Hence (3) is true.	9. (1) and (2) have been proved untrue.

152. Proof by reductio ad absurdum. The proof of the preceding theorem is an example of a method of proof which has long had the Latin name *reductio ad absurdum.* Frequently such a proof is called *indirect.*

The two names, however, are not interchangeable. For example, in Theorem 36 line KR was proved parallel to AB by showing that it coincided with KM. This proof is indirect but is not a proof by reductio ad absurdum. An indirect proof consists in showing that any figure which fulfills the requirements of the theorem must necessarily be the one mentioned in the hypothesis.

One essential feature of a proof reductio ad absurdum consists in making all the assumptions possible under the hypothesis (usually three) and then disproving the truth of all but one. Another essential feature of the method is the manner of disproving each of the false assumptions made. This consists in reasoning from the assumption to a conclusion which is seen to be false because it contradicts some statement of the hypothesis, some obvious fact, or some previously established theorem. Of course this result disproves the truth of the assumption, for if correct reasoning leads from an assumption to a conclusion which is untrue, then the assumption on which the reasoning is based is untrue.

Such a method is conclusive, if we make all the assumptions both true and false which are possible under the given hypothesis. Of these we must know beforehand that *one must be true* and *only one can be true*. If, then, we prove all of the assumptions false but one, that one must be true.

An indirect proof has certain advantages. Often it is shorter than a direct proof and occasionally it is easier to obtain. While a direct proof for Theorem 43 is possible, it would be longer than that given in the text, and probably the only way of solving Ex. 102 by the theorems of Book I is a proof by reductio ad absurdum.

Query 1. What theorems, if any, before Theorem 43 are proved by reductio ad absurdum?

Query 2. What theorems of Book I have been proved by superposition? In what other theorems has Postulate II been used?

Query 3. What is meant by reasoning in a circle? Give an example.

Exercises on Inequalities

173. If the diagonals of a parallelogram are unequal, it is not a rectangle.

174. If two sides of a triangle are unequal and the median to the third side is drawn, the angles formed with the base will be unequal.

175. Prove Theorem 41 by the indirect method.

Hint. AC equals BC, is less than BC, or is greater than BC. Then use Theorem 39.

176. In the triangle ABC, if K is any point in AB and R any point in BC, prove that AB plus BC is greater than AK plus KR plus RC.

177. The difference between two sides of a triangle is less than the third side.

178. If one side of a triangle is divided into two parts by a perpendicular from the opposite vertex, each part of that side is less than the adjacent side of the triangle.

179. AB and BC are equal sides of the triangle ABC. If BC is produced to the point K and KA is drawn, the angle BAK is greater than the angle BKA.

180. Any point (except the vertex) in one of the equal sides of an isosceles triangle is unequally distant from the extremities of the base.

181. ABC is a triangle in which KB and KC bisect the angles B and C respectively. Show that if AB is greater than AC, KB is greater than KC.

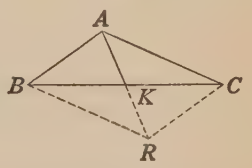
182. In the quadrilateral $ABCD$, AD is the longest and BC the shortest side. Prove that the angle B is greater than the angle D and that the angle C is greater than the angle A.

183. If in the triangle ABC the angle A is bisected by a line meeting BC at K, AB is greater than BK and AC is greater than CK.

HINT. Compare the angles at K with the two at A.

184. If lines are drawn from any point within a triangle to the extremities of one side, they will form an angle greater than the angle formed by the other two sides.

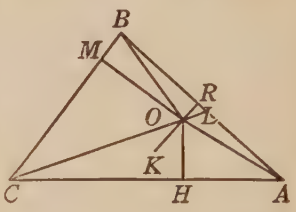
185. AK is a median of triangle ABC. Prove that AB plus AC is greater than twice AK.



HINT. Extend AK to R, making KR equal to AK, and draw BR and CR. Then study the triangle ABR, or the triangle ACR.

186. $ABCD$ is a parallelogram. K, R, L, and M are points on AB, BC, CD, and DA respectively such that AK equals CL and BR equals DM. Prove that $KRLM$ is a parallelogram.

153. Geometric fallacies. Many fallacious geometric proofs exist in which the error is fairly well concealed. One of the best known of these is the proof that *any* triangle is isosceles.



Let $\triangle ABC$ be any triangle. Draw CL the bisector of $\angle ACB$ and let KR be $\perp$ to AB at its mid-point R. Call the intersection of these two lines O. Then draw OH and $OM \perp$ respectively to AC and BC.

Then $\triangle OCM$ is congruent to $\triangle OCH$.			§ 50
Therefore	$HO = OM$,		
and	$HC = MC$.	(1)	§ 27
Now	$AO = BO$.		§ 116
Therefore	$\triangle AOH = \triangle BOM$.		§ 97
Therefore	$AH = BM$.	(2)	§ 27
(1) + (2),	$AH + HC = BM + MC$, or $AC = BC$.		

Therefore any triangle is isosceles.

QUERY. Where is the error in the foregoing?

HISTORICAL NOTE. The earliest Greek textbook on mathematics was written by Hippocrates of Chios about 440 B.C. There were many other writers of elements of geometry before Euclid, but his work superseded theirs and held the field alone. Nearly seven centuries after Euclid, Theon of Alexandria, who was the father of Hypatia, the heroine of Kingsley's novel, prepared an edition of the "Elements" of Euclid for use in his own classes. On the destruction of Alexandria and its University, A.D. 640, some of the learning centered there was preserved in various Syrian cities on the east coast of the Mediterranean. There the Arabs of Bagdad became acquainted with many of the products of Greek culture. The translation of Euclid's "Elements" into Arabic was begun in the reign of Harun-al-Rashid (786–809)

and completed under Al Mamun (813–833). After the Arabs had conquered Spain in 711, among other treasures they brought Euclid's "Elements" with them, and here for over three centuries they guarded it jealously. About the year 1120, however, Adelhard of Bath, an English monk studying in Spain, succeeded in translating the "Elements" from Arabic into Latin. Courses in Euclid's "Elements" soon found a place in many universities of Europe, — especially in those of Germany and Italy, — but it was not until after the invention of printing that the study of Euclid became common in the higher schools. Ten years before Columbus discovered America a Latin version of the "Elements" was published in Venice. By the year 1516 five different editions had been printed. One of these was at Paris. When Constantinople fell into the hands of the Turks in 1453, many educated Greeks fled to Italy, taking with them numerous valuable manuscripts of Greek literature — Euclid's "Elements" among them. Thus it happened that the Paris edition which has been mentioned contained the commentaries of Theon and Hypsicles. The first translation of the "Elements" into English was made in 1570 by Sir Henry Billingsley.

The Germans modified the "Elements" considerably, as did also the French. The English, however, have adhered to Euclid's "Elements" very closely. In this respect the United States has followed the lead of the English.

Euclid's "Elements" was first used at Harvard about 1726 and at Yale about six years later.

The axiom given in § 45 is essentially the famous parallel axiom of Euclid. Many mathematicians have attempted to regard this axiom as a theorem and obtain a proof of it based on Euclid's other axioms. Some, Legendre among them, thought they had obtained valid proofs, but all the attempts at proof, when carefully examined, were found to assume at some point the truth of the axiom itself. Indeed, the mere fact that Euclid was acute enough not to place this axiom with the others is regarded by some as enough evidence in itself to stamp him as a great mathematician.

At the present time no well-informed person attempts to prove this axiom. Correct notions in regard to it were arrived at independently by three men: Gauss (1777–1855), Bolyai (1802–1860), and Lobachevsky (1793–1856). Lobachevsky set himself, not the problem of proving the axiom, but of finding out what would result if instead of it he assumed the following: Through a point outside a line *more than one* parallel to

the line can be drawn. Leaving the other axioms of Euclid unchanged, he was then able to develop a perfectly consistent system of geometry very different from that of Euclid. His system is consistent because it leads to no contradictory conclusions — to no two theorems which make precisely opposite assertions. It does, however, lead to theorems which contradict theorems of Euclidean geometry. For example, one theorem of non-Euclidean geometry asserts: The sum of the angles of any triangle is less than two right angles (see § 66). Moreover, in Lobachevsky's geometry similar triangles do not exist.

Later Riemann replaced Euclid's axiom by the following: Through a point outside a line *no* parallel to the line can be drawn. He then developed a system of geometry different from that of both Euclid and Lobachevsky.

All this may be somewhat confusing, but a proper conception of what mathematics really is will help to clear up the difficulty which exists. A distinguished American mathematician defined mathematics as the science which draws necessary conclusions. This implies, of course, a body of conclusions drawn from certain fundamental assumptions, which do not necessarily deal with numbers, as in arithmetic and algebra, or with space, as in geometry. The simple fact is that Lobachevsky changed one of Euclid's assumptions, and consequently many of the necessary conclusions were different from those of Euclid.

BOOK II

THE CIRCLE

154. Circle. A *circle* is a closed plane curve every point of which is equally distant from a point in the plane of the curve.

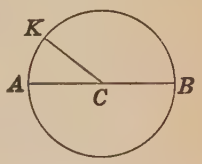
155. Center of a circle. The *center of a circle* is the point in its plane which is equally distant from every point of the circle.

156. Radius. A *radius* of a circle is a straight line from the center to the circle.

157. Diameter. A *diameter* is a straight line through the center terminating in the circle at both ends.

158. Arc. An *arc* is any continuous portion of a circle.

In the adjacent figure the entire curved line is a circle, C is the center, CK a radius, AB a diameter, and KB an arc.



159. Circumference. The length of a circle is the *circumference.*

Some of the more useful inferences from the preceding definitions will now be stated.

160. A diameter is equal to the sum of two radii.

161. Equal circles are congruent.

162. All radii of the same circle or of equal circles are equal.

163. All diameters of the same circle or of equal circles are equal.

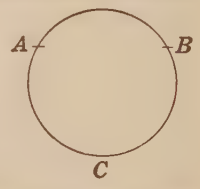
164. Circles having equal radii or equal diameters are equal.

165. If the centers of two equal circles which lie in the same plane be made to coincide, the circles will coincide.

166. Semicircle. A *semicircle* is an arc which is half the circle.

167. Minor arc. An arc less than a semicircle is called a *minor arc.*

168. Major arc. An arc greater than a semicircle is called a *major arc.*



The word *arc* alone will be understood to mean either a major or a minor arc. To avoid all ambiguity it is customary to denote a minor arc by two letters and a major arc by three. Thus, in the adjacent figure the arc AB denotes the minor arc and arc ACB denotes the major arc.

169. Central angle. A *central angle* is an angle whose vertex is at the center of a circle and whose sides are two of its radii.

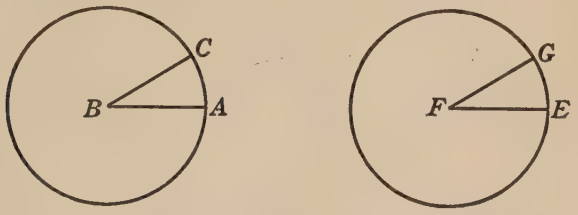
170. Intercept and subtend. An angle is said to *intercept* the arc cut off between its sides, and the arc is said to *subtend* the angle.

Two parallel lines may also be spoken of as *intercepting* two arcs of a circle.

NOTE. The use of the terms *circle* and *circumference* has long been confusing. Both words have been used to designate the curve itself. The word *circle* has been and still is used to denote both the curved line itself and the portion of the plane bounded thereby. When the latter is meant we shall use the phrase *the area of the circle.*

Theorem 1

171. *In the same circle or in equal circles if two central angles are equal, their intercepted arcs are equal.*



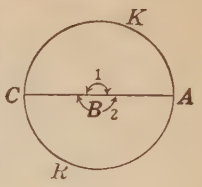
Given the two equal circles B and F in which the central angles B and F are equal and the intercepted arcs are AC and EG respectively.

To prove *that the arc AC equals the arc EG.*

Proof

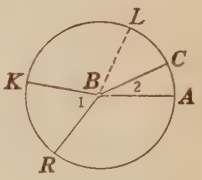
1. If $\angle B$ and $\angle F$ are equal and in the same circle, let one angle rotate about the center until the two coincide. If $\angle B$ and $\angle F$ are in different circles, place the circle with center F on the circle with center B so that the point F falls on the point B and the sides of $\angle F$ fall on the sides of $\angle B$.	1. By hypothesis $\angle B = \angle F$, and § 20.
2. Then E falls on A and G falls on C.	2. § 162.
3. No point of arc GE can fall outside or inside of arc CA.	3. § 165.
4. Therefore arc EG coincides with arc AC and is equal to it.	4. Why?

172. ***Corollary 1.*** *Any diameter of a circle bisects it.*



Proof. Let B be the center and AC any diameter. Then the straight angle ABC = the straight angle ABC; that is, $\angle 1 = \angle 2$. Therefore arc AKC = arc ARC (§ 171).

173. ***Corollary 2.*** *In the same circle or in equal circles, if two central angles are unequal, the greater angle intercepts the greater arc.*



Hints. Let $\angle 1$ be greater than $\angle 2$. Lay off $\angle ABL$ equal to $\angle 1$. BL falls outside $\angle ABC$. Why? Arc AL = arc KR. Why? Conclusion?

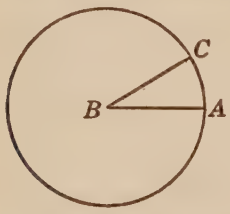
Exercises

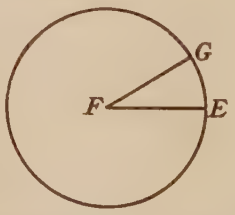
1. Two intersecting diameters divide a circle into two pairs of equal arcs.

2. Two perpendicular diameters divide a circle into four equal arcs.

Theorem 2

174. *In the same circle or in equal circles, if two arcs are equal, they subtend equal central angles.*





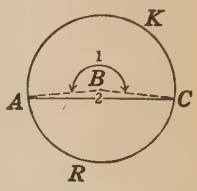
Given two equal circles *B* and *F* in which the arc *AC* equals the arc *EG*.

To prove *that* $\angle F = \angle B$.

Proof

1. Place the circle with center F on the circle with center B so that FE will coincide with its equal BA, the point F falling on B, and the point E on the point A, and make the arc EG fall on the arc AC.	1. §§ 20 and 165.
2. The point G will fall on the point C.	2. Hypothesis.
3. Therefore GF coincides with CB.	3. § 19.
4. Therefore $\angle F = \angle B$.	4. Why?

175. ***Corollary 1.*** *If a line bisects a circle, it is a diameter.*



Proof. Let a straight line CA make arc AKC = arc ARC. Now suppose the center B is not on AC and draw BC and BA. Then since arc CKA = arc CRA, by § 174 one angle CBA = the other angle CBA; that is, $\angle 1 = \angle 2$. Therefore $\angle CBA$ is a straight angle, and hence the line CBA is a straight line; that is, the center B must lie on AC. Therefore AC is a diameter.

176. ***Corollary 2.*** *In the same circle or in equal circles, if two arcs are unequal, the greater arc subtends the greater central angle.*

HINT. Prove in a manner like that of § 173.

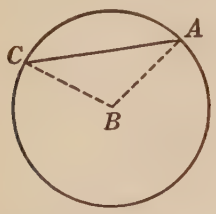
177. Chord. A *chord* is a straight line whose extremities are on the circle.

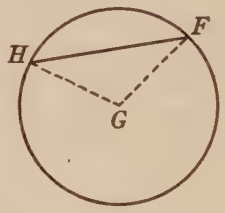
A chord is said to *subtend* an arc. Every chord, except a diameter, subtends two unequal arcs, a minor arc and a major arc. If neither is specified, the minor arc will be the one meant.

QUERY. Is a diameter a chord?

Theorem 3

178. *In the same circle or in equal circles, if two arcs are equal, their chords are equal.*





Given, in the equal circles whose centers are respectively *B* and *G*, the chords *AC* and *FH*, subtending the equal arcs *AC* and *FH* respectively.

***To prove** that* *chord AC = chord FH.*

Proof

1. Draw the radii AB, CB, FG, and HG.	1. § 19.
2. Arc AC = arc FH.	2. Hypothesis.
3. In $\triangle ABC$ and FGH, $\angle B = \angle G$.	3. § 174.
4. $BA = GF$ and $BC = GH$.	4. § 162.
5. $\triangle ABC$ is congruent to $\triangle FGH$.	5. Why?
6. Therefore chord AC = chord FH.	6. Why?

Exercises

3. If three diameters, AB, FG, and KR, divide a circle into six equal arcs, prove that the six angles at the center contain 60° each.

4. A, K, B, and R are points in order on a circle with center C such that arc AK equals arc BR. Prove that the angle KCR equals the angle ACB.

5. AB is a chord of a circle equal to its radius. Prove that the arc AB is one sixth of the whole circumference.

6. If the arcs AB, BC, CD, and DE of the same circle are equal, then the chords AC, BD, and CE are equal.

7. K, R, and L are three points on a circle such that arcs KR, RL, and LK are equal. The chords KR, RL, and LK are drawn. Prove that the triangle KRL is equiangular.

Theorem 4 (*Converse of Theorem 3*)

179. *In the same circle or in equal circles, if two chords are equal, their subtended arcs are equal.*

Given (using the figure of § 178) the equal circles whose centers are respectively *B* and *G*, and chord *AC* equal to chord *FH*.

To prove *that* $\text{arc } AC = \text{arc } FH$.

HINTS. $\triangle ABC$ is congruent to $\triangle FGH$. Why?

$\angle B = \angle G$. Why?

$\text{Arc } AC = \text{arc } FH$. § 171

QUERY 1. If two circles are not equal, would equal chords subtend arcs equal in length? Explain.

QUERY 2. If two circles are not equal, would arcs equal in length have equal chords? Explain.

EXERCISES

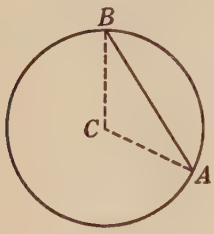
8. If chords AB, BC, CD, and DE are equal, then chords AC, BD, and CE are equal.

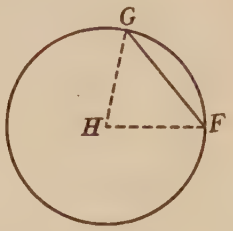
9. A, K, B, and R are points in order on a circle such that the chord AB equals the chord KR. Prove that the arc AK equals the arc BR.

10. Two diameters of a circle are perpendicular to each other. Prove that the chords joining their extremities form a square.

Theorem 5

180. *In the same circle or in equal circles, if two minor arcs are unequal, the greater arc has the greater chord.*





Given the equal circles whose centers are respectively *C* and *H*, in which minor arc *AB* is greater than minor arc *FG*, and chords *AB* and *FG*.

To prove that *chord $AB >$ chord FG.*

Proof

1. Draw radii AC, BC, FH, and GH.	1. § 19.
2. In $\triangle ABC$ and FGH, $CA = HF$, and $CB = HG$.	2. Why?
3. $\angle C > \angle H$.	3. § 176.
4. Therefore chord $AB >$ chord FG.	4. § 150.

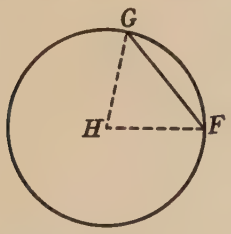
Exercises

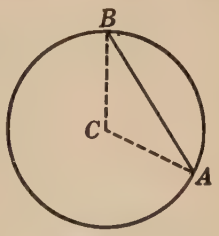
11. AK is a diameter and KB and KC are chords such that KB lies between AK and KC. Prove that the chord KB is greater than the chord KC.

12. A diameter of a circle is greater than any other chord of the circle.

Theorem 6 *(Converse of Theorem 5)*

181. *In the same circle or in equal circles, if two chords are unequal, the greater chord has the greater minor arc.*





Given the equal circles whose centers are respectively *C* and *H*, with chord *AB* greater than chord *FG*.

To prove *that* $\text{arc } AB > \text{arc } FG$.

Proof

1. Draw radii AC, BC, FH, and GH.	1. § 19.
2. In $\triangle ABC$ and FGH, $AC = FH$.	2. Why?
3. $BC = GH$.	3. Why?
4. Chord $AB >$ chord FG.	4. Hypothesis.
5. $\angle C > \angle H$.	5. § 151.
6. Therefore $\text{arc } AB > \text{arc } FG$.	6. § 173.

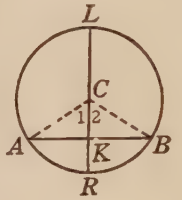
EXERCISES

13. AB is a diameter and BR and BK are chords with BK the greater. Prove chord AK is less than chord AR.

14. R is the center and AB, BC, and CA are chords of a circle. If AB is greater than BC and BC is greater than CA, prove the angle ARC less than 120°.

Theorem 7

182. *If a line passes through the center of a circle and is perpendicular to a chord, it bisects the chord and the arcs subtended by it.*



Given *LR* passing through *C*, the center of the circle, and perpendicular to chord *AB* at *K*.

To prove *that* $AK = KB$, *arc* $AR =$ *arc* BR, *and arc* $AL =$ *arc* BL.

Proof

1. Draw the radii AC and BC.	1. § 19.
2. In the rt. $\triangle AKC$ and BKC, $CK = CK$ and $CA = CB$.	2. Why?
3. $\triangle AKC$ is congruent to $\triangle BKC$.	3. Why?
4. $AK = KB$.	4. Why?
5. Also $\angle 1 = \angle 2$.	5. Why?
6. arc AR = arc RB.	6. § 171.
7. $\angle ACL = \angle BCL$.	7. Why?
8. Therefore arc AL = arc BL.	8. § 171.

Exercises

15. If a diameter bisects a chord, it is perpendicular to the chord and bisects its arcs.

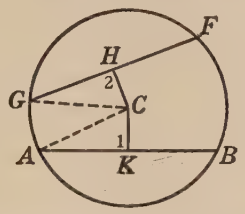
Hint. In a figure like that of § 182, prove the angles at K equal.

16. If a diameter bisects an arc, it bisects the chord of the arc at right angles.

17. If a line bisects a chord and one of its arcs, it is perpendicular to the chord, passes through the center, and bisects the other arc.

Theorem 8

133. *In the same circle or in equal circles, if two chords are equal, they are equally distant from the center.*



Given the chord *GF* equal to the chord *AB* in circle *C*, and *CH* and *CK* their respective distances from the center.

To prove *that* $CH = CK$.

Proof

1. Draw radii CG and CA.	1. § 19.
2. In $\triangle AKC$ and GHC, $\angle 1 = \angle 2 = 1$ rt. $\angle$.	2. § 120.
3. Chord AB = chord GF.	3. Hypothesis.
4. $GH = \frac{1}{2} GF$ and $AK = \frac{1}{2} AB$.	4. § 182.
5. $GH = AK$.	5. Why?
6. $CG = CA$.	6. Why?
7. $\triangle AKC$ is congruent to $\triangle GHC$.	7. Why?
8. Therefore $CH = CK$.	8. Why?

Exercises

18. AB and AC are equal chords and AK is a diameter. Prove that the angle BAK is equal to the angle CAK.

19. The arcs AB, BC, and CA of a circle are equal. Show that the distance of the chords AB, BC, and CA from the center is one half the radius.

Theorem 9

184. *In the same circle or in equal circles, if two chords are equally distant from the center, they are equal.*

Given (using the figure of § 183) the chords *AB* and *FG* with their respective distances from the center, *CK* and *CH*, equal.

***To prove** that* *chord AB = chord FG.*

HINT. Prove that $\triangle ACK$ is congruent to $\triangle GCH$.

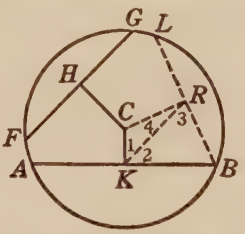
EXERCISES

20. Two chords from one extremity of a diameter making equal angles with it are equal.

21. If the perpendiculars from the center upon two chords are equal, the major arcs subtended by the chords are equal.

Theorem 10

185. *In the same circle or in equal circles, if two chords are unequal, the greater is at the less distance from the center.*



Given in the circle *C* the chord *AB* greater than the chord *FG*, and their respective distances from the center, *CK* and *CH*.

***To prove** that* *$CK < CH$.*

Proof

1. Let BL be a chord equal to the chord FG and let CR be $\perp$ to BL. Join K and R.	1. § 42.
2. $KB = \frac{1}{2} AB$ and $BR = \frac{1}{2} BL$.	2. § 182.

3.	Chord $AB >$ chord FG or its equal, chord BL.	3. Hypothesis.
4.	$KB > BR$.	4. § 140.
5.	$\angle 3 > \angle 2$. (1)	5. § 145.
6.	$\angle 3 + \angle 4 = \angle 2 + \angle 1$. (2)	6. § 40.
7.	From (1) and (2), $\angle 4 < \angle 1$.	7. § 142.
8.	$CK < CR$.	8. § 148.
9.	But $CR = CH$.	9. § 183.
10.	$CK < CH$.	10. Why?

QUERY. Does Theorem 10 hold when one chord is a diameter?

EXERCISES

22. The arcs AB, BC, and CD of a circle are equal. K is the mid-point of the arc AB, KR is perpendicular to the chord AB at R, and CH is perpendicular to the chord BD at H. Prove that CH is greater than KR.

23. AB is a diameter, G the center of a circle, and AK and AR chords such that K lies between B and R. The line GH is perpendicular to AK at H, and GL is perpendicular to AR at L. The line HL is drawn. Prove that the angle LHG is greater than the angle GLH.

Theorem 11

186. *In the same circle or in equal circles, if two chords are unequally distant from the center, the chord at the greater distance is the less.*

Given (using the figure of § 185) chords *FG* and *AB*, with distance *CH* greater than distance *CK*.

To prove that *chord $FG <$ chord AB.*

HINTS.	$CR > CK$.	Why?
	$\angle 1 > \angle 4$.	Why?
	$\angle 2 < \angle 3$.	Why?
	$BR < BK$, etc.	Why?

QUERY. Does Theorem 11 hold when one chord passes through the center?

Exercises

24. AB is a diameter, O the center, and BK and BR chords of a circle. LO is perpendicular to BK at L and OH is perpendicular to BR at H. If OL is less than OH, prove that chord AK is less than chord AR.

25. If in the same circle or in equal circles two chords are unequally distant from the center, the chord at the greater distance subtends the less minor arc.

26. If two chords of a circle bisect each other, they are diameters.

Exercises on Chords, Arcs, and Central Angles

27. Two intersecting chords making equal angles with the diameter passing through their point of intersection are equal.

28. If a diameter is drawn through a point within a circle equidistant from two points upon it, the arc between the two points is bisected.

29. If a circle is divided into six equal arcs, any chord joining adjacent points of division is equal to the radius of the circle.

30. If a line bisects an arc and is perpendicular to the chord of the arc, it will, if produced, pass through the center of the circle.

31. Two points, each equidistant from the ends of a chord, determine a line passing through the center of the circle.

32. A is the center of a circle and K is a point outside. KRL and KHG make equal angles with AK and cut the circle in R and L and in H and G respectively. Prove chords RL and HG equal.

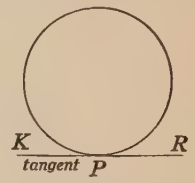
33. B is the mid-point of the arc AC of a circle, and BK and BR are drawn perpendicular to the radii OA and OC respectively. Prove that BK equals BR.

34. If from the extremities of any diameter perpendiculars are drawn upon any chord (produced if necessary), the feet of the perpendiculars are equidistant from the center of the circle.

35. If two unequal chords intersect in a circle, the greater chord makes the less acute angle with the diameter through the point of intersection.

36. A and B are the centers of two circles which intersect in K and R. KL is drawn to the mid-point of AB. A line through K perpendicular to KL cuts one circle in H and the other in G. Prove that the chord KH equals the chord KG.

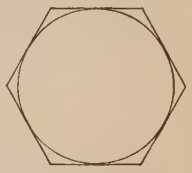
187. Tangent. A *tangent* to a circle is a straight line which, however far it may be produced, has only one point in common with the circle.



If a straight line is tangent to a circle, the circle is also tangent to the line.

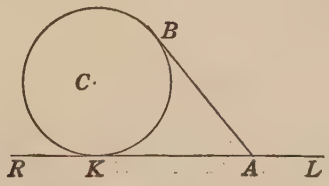
The point common to the circle and to the tangent is called the *point of tangency* or *point of contact.*

Thus, in the accompanying figure, KR is the tangent and P the point of contact.



188. Circumscribed polygon. A polygon whose sides are tangent to a circle is called a *circumscribed polygon.*

189. Tangent from an outside point. A tangent to a circle from an outside point means a *line-segment* whose extremities are the outside point and the point of contact.

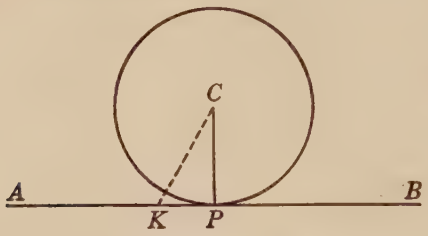


In the above figure the line-segments AB and AK are tangents to the circle from the outside point A.

190. Common tangent. A *common tangent* to two circles is a straight line tangent to each.

Theorem 12

191. *If a line is perpendicular to a radius at its outer extremity, it is tangent to the circle.*



Given the circle *C* and *AB* perpendicular to the radius *CP* at *P*, which is on the circle.

To prove *that* *AB is tangent to the circle.*

Proof

1. Draw CK, where K is any point on AB except P.	1. § 19.
2. $CK > CP$.	2. § 149.
3. K is outside the circle.	3. Why?
4. Hence any point of AB except P is outside the circle, and AB is tangent to the circle.	4. § 187.

Exercises

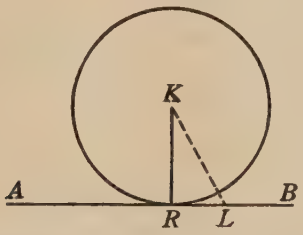
37. $ABCDE$ is a circumscribed equiangular pentagon. OK and OR are radii to the points of contact of sides AB and AE. How many degrees in the angle KOR? (Solve after proving Theorem 13.)

38. Perpendiculars to a diameter at its extremities are parallel tangents.

Query. Can either of two tangents to a circle from an outside point be perpendicular to the chord joining the points of contact?

Theorem 13 *(Converse of Theorem 12)*

192. *If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of tangency.*



Given the circle *K*, the line *AB*, tangent to it at *R*, and radius *KR*.

To prove that *KR* is ⊥ to *AB*.

Proof

1. Draw *LK* to *L*, any point on *AB* except *R*.	1. § 19.
2. $LK > KR$.	2. §§ 154 and 187.
3. Hence *KR* is the shortest line from *K* to *AB*, and *KR* is ⊥ to *AB*.	3. § 149.

QUERY. To how many radii of a circle can one tangent be perpendicular?

EXERCISES

39. Tangents to a circle at the extremities of a diameter are parallel to each other.

40. Tangents to a circle at the extremities of two perpendicular diameters meet in *G*, *H*, *L*, and *M*. Prove that *GHLM* is a circumscribed square.

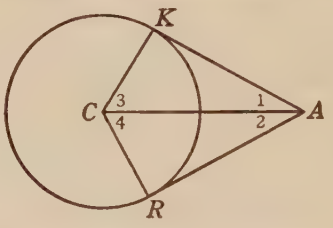
41. If two tangents to a circle are parallel, the points of contact and the center are in a straight line.

Theorem 14

193. *If two tangents are drawn to a circle from an outside point,*

(1) *the tangents are equal;*

(2) *the line joining the outside point to the center bisects the angle between the tangents and the angle between the radii drawn to the points of contact.*



Given *AK* and *AR* tangents from the point *A* outside the circle whose center is *C*, the line *AC*, and the radii *CK* and *CR* to the points of contact *K* and *R*.

To prove that (1) $AK = AR$.
(2) $\angle 1 = \angle 2$ *and* $\angle 3 = \angle 4$.

Proof

1. In $\triangle AKC$ and ARC,	$\angle K = \angle R$.	1. Why?
2.	$CA = CA$.	2. Why?
3.	$CK = CR$.	3. Why?
4. $\triangle AKC$ is congruent to $\triangle ARC$.		4. Why?
5. Therefore $AK = AR$, $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$.		5. Why?

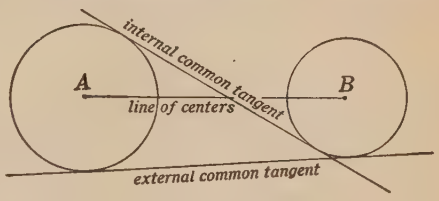
EXERCISES

42. Two circles have two common tangents which do not pass between them. The points of tangency are A and B for one tangent and C and D for the other. Prove that AB equals CD.

43. The line AB touches a circle at K, the line AL at L, and the line BR at R. Prove that AB equals the sum of AL and BR.

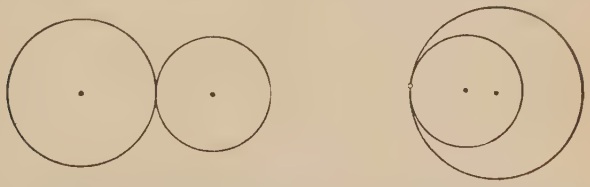
44. In a circumscribed quadrilateral the sum of two opposite sides equals the sum of the other two.

194. Line of centers. The line-segment joining the centers of two circles is called the *line of centers.*



195. Tangent circles. If two circles have only one point in common, each is tangent to the other.

196. Externally tangent circles. If each of two tangent circles is outside the other, they are *tangent externally.*



197. Internally tangent circles. If one of two tangent circles is within the other, they are *tangent internally.*

198. Internal common tangent. A common tangent of two circles which passes between them is an *internal common tangent.*

199. External common tangent. A common tangent which does not pass between two circles is an *external common tangent.*

QUERY 1. What is the greatest and the least number of common tangents two circles can have?

QUERY 2. What other possible numbers of common tangents to two circles are there?

QUERY 3. What are the relative positions of the two circles in each of the preceding cases?

QUERY 4. Can two circles have more than two, less than two, or no common external tangents?

QUERY 5. What are the relative positions of the two circles in each of these cases?

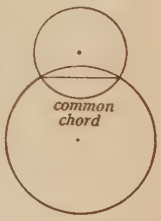
QUERY 6. Change *external* to *internal* and answer the two preceding queries.

QUERY 7. Can equal circles be tangent externally? internally?

EXERCISES

45. The centers of two unequal circles and the point of intersection of their two external common tangents are in the same straight line.

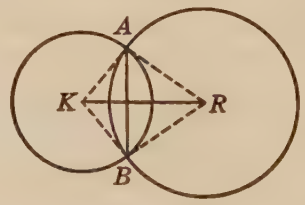
46. The centers of two circles and the point of intersection of their two internal common tangents are in the same straight line.



200. Common chord. The *common chord* of two intersecting circles is the chord joining the points of intersection.

Theorem 15

201. *If two circles intersect, the line of centers bisects their common chord at right angles.*



Given the circles *K* and *R* intersecting at *A* and *B*, the line of centers *KR*, and the common chord *AB*.

To prove *that KR bisects AB at right angles.*

Proof

1. Draw the radii KA, KB, RA, and RB.	1. § 19.
2. Then K is equally distant from A and B.	2. Why?
3. R is also equidistant from A and B.	3. Why?
4. Therefore KR is the mid-perpendicular of AB.	4. § 118.

QUERY 1. If one of three unequal circles is tangent to each of the other two, how many points of contact must there be?

QUERY 2. How many may there be?

QUERY 3. What are the relative positions of the circles in each case?

QUERY 4. Has Theorem 15 more than one converse? If so, state each.

QUERY 5. Of the fifteen theorems on circles, which have more than one converse?

EXERCISES

47. Three circles pass through two points K and R. Prove that their centers are in a straight line.

48. A is the center of a circle, K is a point on the radius AB. With K as the center and KB as the radius another circle is described. Prove that the two circles are tangent.

HINT. Suppose the circles intersect at a point H, distinct from B. Draw the mid-perpendicular of BH. Can this mid-perpendicular pass through both centers?

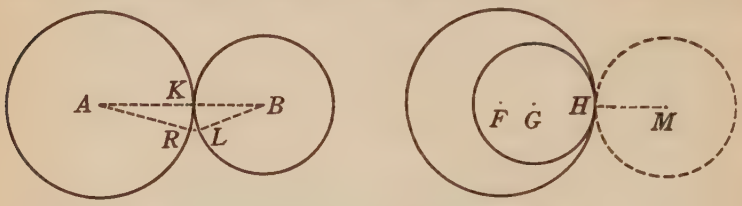
49. A is the center of a circle. The radius AB is produced to C. With C as the center and CB as the radius another circle is described. Prove that the two circles are tangent.

HINT. Draw KB perpendicular to AB and show that all points of circle A except point B are on one side of KB and all points of circle C except point B are on the other side of it.

50. Prove the theorem of Ex. 49, using the hint of Ex. 48.

Theorem 16

202. *If two circles are tangent externally or internally, the centers and the point of tangency are in a straight line.*



Case I. *When the circles are tangent externally.*

Given the circles *A* and *B* tangent externally at *K*.

To prove *that the points A, K, and B are in a straight line.*

Proof

1. Suppose the line of centers does not pass through K, but cuts the circle A in R and the circle B in L. Draw the radii AK and BK and the line LR.	1. § 19.
2. Now $AK = AR$.	2. Why?
3. $BK = BL$.	3. Why?
4. $AK + KB < AR + RL + LB$; that is, AKB is shorter than any other line from A to B by the portion RL.	4. Why?
5. Therefore A, K, and B are in the same straight line.	5. § 146.

Case II. *When the two circles are tangent internally.*

Given the circles *F* and *G* tangent internally at *H*.

To prove *that the points F, G, and H are in a straight line.*

1. Suppose the circle M is tangent to the circle F at H. Draw HM.	1. Why?

2. Circle M will be tangent to circle G.	2. § 195.
3. Then points G, H, and M are in the straight line HM.	3. Case I.
4. Also points F, H, and M are in the straight line HM.	4. Case I.
5. F, G, and H are all in the same straight line.	5. Steps 3 and 4.

203. ***Corollary 1.*** *The line of centers of two externally tangent circles is equal to the sum of the two radii.*

204. ***Corollary 2.*** *The line of centers of two internally tangent circles is equal to the difference of the two radii.*

205. ***Corollary 3.*** *A line perpendicular to the line through the centers of two tangent circles at their point of contact is the common tangent of the two circles.*

Exercises

51. If two circles are tangent externally, two tangents drawn to them from any point in their common internal tangent are equal.

52. If two circles are tangent externally, their common internal tangent bisects the portion of their common external tangent between the points of contact.

206. Concentric circles. Two or more circles which have the same center are called *concentric circles.*

Exercises on Tangents and Tangent Circles

53. AB and AC are tangent to a circle at B and C respectively. Prove that AK, a line from A to the center, is the mid-perpendicular of BC.

54. AB and AC are two tangents to a circle at B and C respectively. A third tangent touching the arc BC cuts AB at K and AC at R. Prove that the perimeter of the triangle AKR is equal to AB plus AC.

55. The segments of two internal common tangents of two circles included between their respective points of contact are equal.

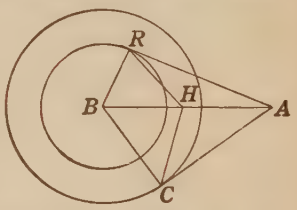
56. Tangents CA and DB of a circle drawn at the extremities of the diameter AB meet a third tangent CD at C and D respectively. K is the center of the circle. Prove that the angle CKD is a right angle.

57. Tangents to a circle at the extremities of any pair of diameters which are not perpendicular to each other form a rhombus.

58. If the opposite sides of a circumscribed quadrilateral are parallel, the four sides are equal.

59. The contact points of the exterior common tangents of two circles are A and B for one circle and C and D for the other. Prove AB parallel to CD.

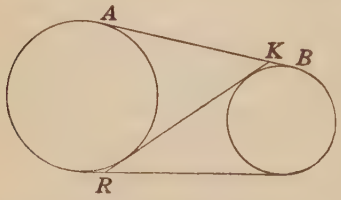
60. Tangents from the same point to two concentric circles cannot be equal.

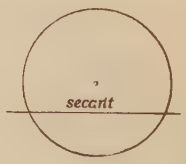


HINT. Let AC and AR be the tangents, B the center, and H the mid-point of AB. Draw HC and HR. Then study triangles AHC and AHR (see Ex. 148 and § 150).

61. If a point lies on each of two circles and also on the line through their centers, the circles are tangent to each other.

62. A and B are the points of contact of a common exterior tangent to two circles. One common interior tangent cuts AB at K and the other common exterior tangent at R. Prove that AB equals KR.





207. Secant. A *secant* is a line which cuts a circle in two points.

Theorem 17

208. *If two arcs of a circle are intercepted by two parallel lines, the arcs are equal.*

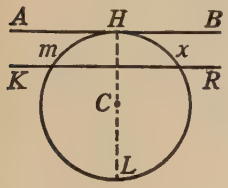


FIG. 1

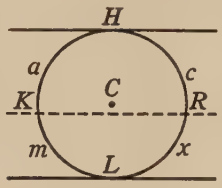


FIG. 2

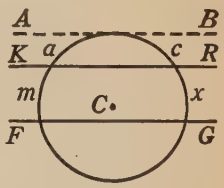


FIG. 3

Case I. *When one of the parallels is a tangent.*

Given (Fig. 1) the circle *C* and *AB*, the tangent at *H*, parallel to the secant *KR*, the two intercepting the arcs *m* and *x*.

To prove *that* *arc m = arc x.*

Proof

1. Draw the diameter *HL*.	1. § 19.
2. Then *HL* is ⊥ to *AB*.	2. § 192.
3. *HL* is ⊥ to *KR*.	3. § 47.
4. Therefore arc *m* = arc *x*.	4. § 182.

Case II. *When both parallels are tangents.*

Given (Fig. 2) the circle *C* and two tangents parallel to each other and touching the circle at *H* and *L* respectively.

To prove *that* *arc HKL = arc HRL.*

Proof

1. Let *KR*, ∥ to one tangent, form the arcs *m*, *x*, *a*, and *c*.	1. § 45.
2. *KR* is ∥ to both tangents.	2. If *KR* cuts one parallel, it would cut both. § 46.
3. Then *m* = *x* and *a* = *c*.	3. Case I.
4. Hence arc *HKL* = arc *HRL*.	4. § 31.

Case III. *When both parallels are secants.*

HINT. Draw $AB \parallel$ to KR, touching the circle and forming the arcs a and c, as indicated in the figure.

QUERY 1. Is the statement which follows a converse of Theorem 17? If two lines intercept equal arcs of a circle, the lines are parallel.

QUERY 2. Is this statement true?

QUERY 3. Has Theorem 17 more than one converse? Illustrate.

QUERY 4. Is the following modification of Theorem 17 true? If in the same circle or in equal circles two arcs are intercepted by parallel lines, the two arcs are equal.

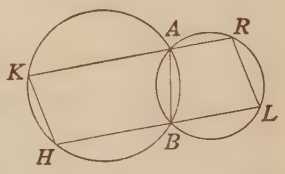
209. Inscribed polygon. A polygon whose vertices lie on a circle is called an *inscribed polygon.*

EXERCISES

63. AD and BC are the nonparallel sides of the inscribed trapezoid $ABCD$. Prove that the arc ABC equals the arc DAB, and prove that the arc ADC equals the arc BCD.

64. The nonparallel sides of an inscribed trapezoid are equal and the diagonals are equal.

65. Two circles intersect in A and B. Through A and B parallels KAR and HBL are drawn, points K and H being on one circle and R and L on the other. Prove that the chords AB, KH, and RL are equal.



66. Two chords of a circle perpendicular to a third at its extremities are equal.

210. Degree of arc. If an arc whose central angle is a right angle is divided into ninety equal parts, one of these parts is called a *degree of arc.*

From the preceding definition and § 56 it follows that *a central angle of one degree* intercepts on the circle *an arc of one degree.*

211. Relation of a central angle to its intercepted arc. The number of angle degrees in a central angle is equal to the number of arc degrees in its intercepted arc.

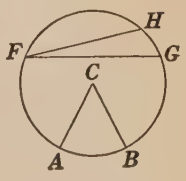
The usual statement of this relation is the theorem

A central angle is measured by its intercepted arc.

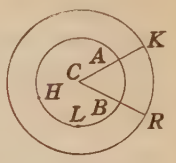
The truth of this theorem will be assumed.

The same number, therefore, expresses the measure of an arc of a circle and of its subtended central angle, but this does not assert that an angle is the same thing as an arc. It merely asserts that they may be equal numerically, just as the number of acres in a field may be equal numerically to the number of rods of fence around the field, without implying that the field and the fence or an acre and a rod are the same thing.

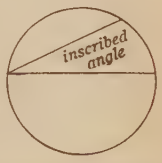
212. Notation for numerically equal arcs and angles. If the last sentence of § 211 is clearly understood, we may write, referring to the adjacent figure in which C is the center, $\angle C = \text{arc } AB$. Later we shall prove that $\angle F$ is measured by one half the number of degrees in the intercepted arc HG, and this statement may be written $\angle F = \frac{1}{2} \text{ arc } HG$. This, however, must be understood to mean: *The number of degrees of angle in angle F equals one half the number of degrees of arc in arc HG.* A similar meaning must be assigned to $\angle C = \text{arc } AB$. In dealing with arcs and angles we shall gain in brevity if the numerical relations which exist are written as equations. This is perfectly correct, since the two members of the equation are equal numbers. Of course the usual laws of operation on equations may then be applied to these statements. The equation form of statement is used in the proofs of Theorems 20, 21, and 23 which follow.



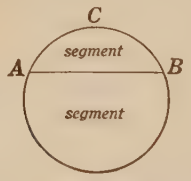
The length of an arc in feet or inches must not be confused with the number of degrees it contains. In the adjacent figure, the angle C is a central angle in both circles. Therefore arc AB and arc KR contain the same number of degrees. But arc AB is shorter than arc KR. In the same circle or in equal circles, however, two arcs of the same number of degrees are of equal length. Thus, if each of arcs AB and HL of the adjacent figure contains $m°$, they are also of equal length in feet or inches.

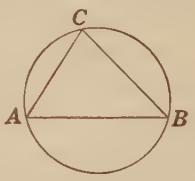


213. Inscribed angle. An angle whose vertex is on a circle and whose sides are chords of the circle is called an *inscribed angle.*



214. Segment. A *segment* of a circle is a figure bounded by an arc of the circle and its chord.





An angle is said to be *inscribed in a segment* or *an arc* if its vertex is on the arc and its sides pass through the ends of the arc.

Thus the angle C may be spoken of as inscribed in the segment ACB or in the arc ACB.

215. Sector. A *sector* of a circle is a figure bounded by two radii and their intercepted arc.

QUERY 1. How many degrees of angle are there about the center of a circle?

QUERY 2. How many degrees of arc are there in a circle?

QUERY 3. What figure may be called either a segment or a sector?

QUERY 4. A chord forms how many segments?

QUERY 5. Two radii form how many sectors?

Exercises

67. The circumference of a circle is 30 ft. How many degrees in an arc of this circle 10 ft. long? in one 4 ft. long?

68. An arc of a circle containing 18° is 3 ft. long. Find the circumference of the circle.

Historical Note. The division of a right angle into ninety degrees is equivalent to dividing a circle into three hundred and sixty degrees. This part of our mathematical heritage, as well as the present division of the day into hours, minutes, and seconds, comes from the Babylonians. Why they adopted such a scheme of division we do not know. Cantor, a great historian of mathematics, suggests that their division of a circle came from their combining an astronomical belief with a geometrical fact. At first, according to his supposition, they supposed that the sun went round the earth in a circle once in three hundred and sixty days, thus making its daily motion $\frac{1}{360}$ of the whole. Moreover, they were familiar with the fact that a chord of a circle equal to the radius has an arc which is exactly one sixth of the circle (§ 348). Such an arc seemed a natural division of the circle, since the sun would take sixty days to pass over it. In some such way they were led to what is called the sexagesimal division of the circle.

About the time the metric system was devised (1790) an attempt was made in France to put the division of a right angle on a decimal (or rather centesimal) basis. The right angle was divided into one hundred *grades*, each grade into one hundred minutes, and each minute into one hundred seconds. A large force of computers, after great expenditure of time and effort, calculated the necessary trigonometric and logarithmic tables to correspond. No attempt has been made to transfer to the centesimal system the many standard reference tables, the countless records of observation, and extensive data of various kinds which workers in mathematics and science use. Until this is done the undoubted merits of the centesimal system will not be apparent. As a consequence, the system has not been used outside of France, and only slightly there.

Although this system and the metric system of measurement, which is closely associated with it, possess great practical advantages over our present or English system of measurement, the difficulty and expense of making the necessary changes in machinery and instruments prevent its adoption.

Theorem 18

216. *An inscribed angle is measured by one half the number of degrees in its intercepted arc.*

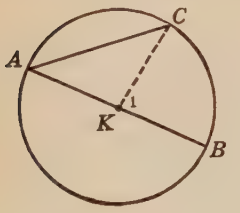


Fig. 1

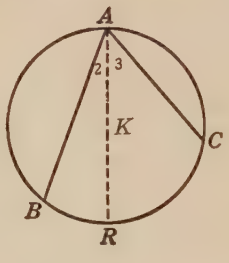


Fig. 2

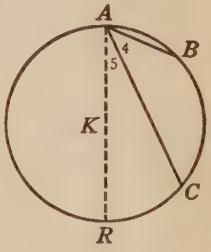


Fig. 3

Given the angle *BAC*, an inscribed angle.

To prove *that $\angle BAC$ is measured by one half the arc BC.*

Proof

Case I. *When the center K is on one side of the angle (Fig. 1).*

1. Draw CK.	1. § 19.
2. Then in $\triangle ACK$, $\angle A = \angle C$.	2. Why?
3. Also $\angle 1 = \angle A + \angle C$.	3. Why?
4. $\angle 1 = 2\angle A$.	4. § 65.
5. But $\angle 1$ is measured by arc BC.	5. § 211.
6. $\angle A$ is measured by one half the arc BC.	6. Since $\angle 1$ is measured by the arc BC, and $\angle A = \frac{1}{2}\angle 1$.

Case II. *When the center K falls within the angle (Fig. 2).*

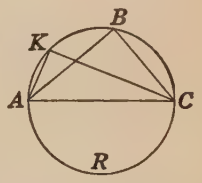
Hints. Draw the diameter AR. $\angle 2$ is measured by what? $\angle 3$ is measured by what? $\angle 2 + \angle 3$, which equals $\angle BAC$, is measured by what? Conclusion?

Case III. *When the center is outside the angle (Fig. 3).*

HINTS. Draw the diameter AR. $\angle BAR$ is measured by what? $\angle 5$ is measured by what? $\angle BAR - \angle 5$, which equals $\angle 4$, is measured by what? Conclusion?

217. ***Corollary 1.*** *An angle inscribed in a semicircle is a right angle.*

HINT. How many degrees in the arc intercepted by the angle?



218. ***Corollary 2.*** *Two angles inscribed in the same segment are equal.*

HINTS. How is $\angle B$ measured? $\angle K$? Conclusion?

219. ***Corollary 3.*** *The opposite angles of an inscribed quadrilateral are supplementary.*

HINTS. If $ABCD$ is an inscribed quadrilateral, what arc measures $\angle a$? $\angle c$? What relation exists between these two arcs? Conclusion?

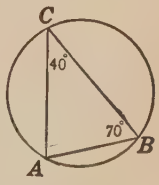
QUERY 1. If a chord bisects an inscribed angle, does it bisect the intercepted arc?

QUERY 2. If a line is drawn from the mid-point of an arc to the vertex of an angle inscribed in that arc, does it bisect the angle?

EXERCISES

69. Two angles of an inscribed triangle are 40° and 70° respectively. Find the number of degrees in each of the arcs subtended by the sides of the triangle.

70. AB, BC, and CA are the sides of an inscribed triangle. The arc AB contains 135°, and the arc BC is twice the arc AC. Find each angle of the triangle.

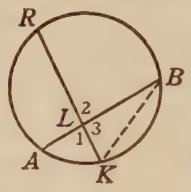


71. Two sides of an inscribed triangle subtend arcs which are one fifth and one eighth of the circle respectively. Find the angles of the triangle.

72. An inscribed angle ABC contains 30°. Prove that the chord AC equals the radius of the circle.

Theorem 19

220. *If two vertical angles are formed by two intersecting chords, each angle is measured by one half the number of degrees in the sum of their intercepted arcs.*



Given the chords *AB* and *KR* intersecting in the point *L*.

To prove *that* $\angle 1$ *and* $\angle 2$ *are each measured by* $\frac{1}{2}$ *(arc* AK $+$ *arc* BR*), and that* $\angle 3$ *and* $\angle ALR$ *are each measured by* $\frac{1}{2}$ *(arc* AR $+$ *arc* BK*).*

HINTS. Draw the chord BK. Then $\angle 1 = \angle 2$. Why?

But $\angle 1 = \angle K + \angle B$. Why?

How is $\angle K$ measured? $\angle B$? Conclusion?

QUERY. If a chord in the figure of § 220 bisects $\angle 1$ and $\angle 2$, does it bisect the arc BR and the arc AK?

EXERCISES

73. If arc AR in the figure of § 220 is 27° and arc BK is 52°, find the number of degrees in $\angle 3$.

74. If arc AR in the figure of § 220 is 30° and $\angle 3$ is 75°, find the number of degrees in the arc KB.

75. If three angles of an inscribed quadrilateral are 70°, 80°, and 110° respectively and an angle between the diagonals is 40°, find the number of degrees in each of the four arcs subtended by the sides.

76. Three angles of an inscribed quadrilateral are 85°, 95°, and 80°. Can these angles be consecutive angles of the quadrilateral in the order given?

77. If two equal chords intersect, the point of intersection divides both in the same way.

Theorem 20

221. *If an angle is formed by a tangent and a chord drawn from the point of contact, the angle is measured by one half the number of degrees in its intercepted arc.*

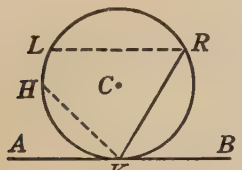


FIG. 1

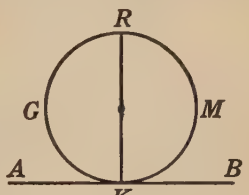


FIG. 2

Given *AB*, a tangent to a circle *C* at *K*, and the chord *KR*.

To prove *that* $\angle BKR$ *is measured by* $\frac{1}{2}$ *arc* KR, *and* $\angle AKR$ *by* $\frac{1}{2}$ *arc* KHR.

Proof

Case I. *When the chord* KR *is not a diameter* (*Fig. 1*).

1. Draw chord $RL \parallel$ to AB.	1. § 45.
2. Then $\angle R = \angle BKR$.	2. Why?
3. Now $\angle R = \frac{1}{2}$ arc KL.	3. Why?
4. Therefore $\angle BKR = \frac{1}{2}$ arc KL.	4. Why?
5. But arc KR = arc KL.	5. Why?
6. Therefore BKR is measured by $\frac{1}{2}$ arc KR.	6. Why?

Now consider the angle AKR.

1. Draw KH so that $\angle AKH$ is less than 90°.	1. AKR is an obtuse angle.
2. Then $\angle AKH = \frac{1}{2}$ arc HK.	2. Why?
3. $\angle HKR = \frac{1}{2}$ arc HR.	3. Why?
4. Therefore $\angle AKH + \angle HKR$, or $\angle AKR$, is measured by $\frac{1}{2}$ arc KHR.	4. Why?

Case II. *When the chord KR is a diameter (Fig. 2).*

1.	Arc $KMR = 180°$.	1. Why?
2.	$\angle BKR = 90°$.	2. Why?
3.	Therefore $\angle BKR$ is measured by $\frac{1}{2}$ arc KMR.	3. Why?

In like manner $\angle AKR$ is measured by $\frac{1}{2}$ arc KGR.

Query 1. By what arcs is an exterior angle of an inscribed polygon measured?

Query 2. If arc KR in Fig. 1 of § 221 is 50°, how many degrees in $\angle AKR$?

Query 3. In Fig. 1 of § 221, if arc LR equals arc KH equals 70°, how many degrees in angle HKR?

Query 4. In Fig. 1 of § 221, if chord LR = chord RK, can you find arc LK? How? $\angle R$? How?

Exercises

78. If a chord of a circle bisects the angle between a tangent and a chord drawn from the point of contact, does it bisect the intercepted arc? Prove.

79. If chords AB and AC of a circle make equal angles with the tangent at A, they are equal.

80. A tangent at the mid-point of an arc of a circle is parallel to the chord of the arc.

81. $ABCD$ is an inscribed polygon. The chords AB, BC, and CD subtend arcs which are one third, one fourth, and one fifth of the circle respectively. Find the number of degrees in each exterior angle of $ABCD$.

82. A, B, and C, three points on a circle, and R, a point within, are the vertices of quadrilateral $ABCR$. Prove that the angle A is less than the exterior angle at C.

83. Two angles of an inscribed triangle are 46° and 62° respectively. Find the arcs subtended by its sides.

Theorem 21

222. *The angle between two secants, or two tangents, or a tangent and a secant, which intersect outside the circle, is measured by one half the number of degrees in the difference of the intercepted arcs.*

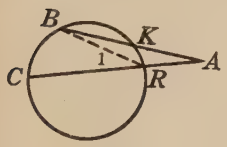


FIG. 1

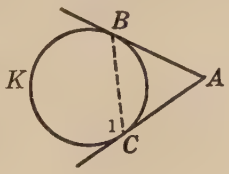


FIG. 2

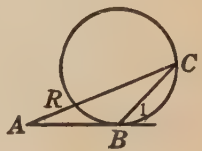


FIG. 3

Given (Fig. 1) the circle *BKR*, and *AKB* and *ARC* two secants intersecting at *A*, a point outside the circle.

To prove *that* $\angle A$ *is measured by* $\frac{1}{2}$ *(arc* BC $-$ *arc* KR*).*

Proof

1. Draw BR, making $\angle 1$ with CR.	1. § 19.
2. Then $\angle 1 = \angle A + \angle B$.	2. Why?
3. Whence $\angle A = \angle 1 - \angle B$.	3. § 51.
4. $\angle 1 = \frac{1}{2}$ arc BC.	4. § 216.
5. $\angle B = \frac{1}{2}$ arc KR.	5. Why?
6. Therefore $\angle A$ is measured by $\frac{1}{2}$ (arc BC $-$ arc KR).	6. § 65.

HINTS. The proof when AB and AC are tangents (Fig. 2) and when one is a tangent and the other a secant (Fig. 3) can be worked out in like manner.

223. ***Corollary 1.*** *The angle between two tangents to a circle is measured by the supplement of the smaller of the two intercepted arcs.*

Outline of proof. In Fig. 2 of § 222 let the minor arc $BC = x°$. Then the major arc $BKC = 360° - x°$. Then, by Theorem 21, $\angle A = \frac{1}{2}[(360° - x°) - x°] = \frac{1}{2}(360° - 2x°) = 180° - x°$.

Exercises

84. If the arc KR (Fig. 1, § 222) is 28° and the arc BC is 86°, find the number of degrees in the angle A.

85. If the arc KR (Fig. 1, § 222) is 28° and the angle A is 65°, find the number of degrees in the arc BC.

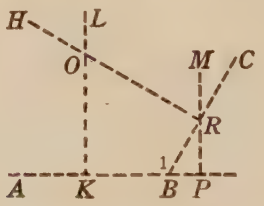
86. If the minor arc BC (Fig. 2, § 222) is one fourth of the major arc BC, find the number of degrees in the angle A.

87. If the angle A (Fig. 3, § 222) contains 46° and the arc BC is five times the arc BR, find the number of degrees in the arc BR and in the arc BC.

88. If a secant bisects the angle between two tangents to a circle, does it bisect each arc intercepted by the tangents?

Theorem 22

224. *If three points are not in the same straight line, one circle and only one can pass through them.*



Given A, B, and C, three points not in the same straight line.

To prove *that one circle and only one can pass through A, B, and C.*

Proof

1. Join A to B and B to C. Let $\angle 1$ be the angle at B, which is less than a straight angle. Let KL be the mid-perpendicular of AB. Draw $MP \perp$ to AB produced, through R, the mid-point of BC.	1. §§ 41, 42.
2. Then KL is $\parallel$ to MP.	2. § 44.
3. $\angle MRB > 90°$.	3. § 144.
4. Now draw HR, the mid-perpendicular of BC. Then HR lies within the angle MRB and intersects KL at some point O.	4. § 46.
5. Then $AO = OB = OC$.	5. § 116.
6. Hence a circle with radius OC and center O will pass through A, B, and C.	6. Previous step.
7. The lines KL and HR intersect in only one point. Therefore but one circle can pass through A, B, and C.	7. Why?

QUERY 1. How many circles can pass through two points?

QUERY 2. Where do all their centers lie?

QUERY 3. Answer the question in Ex. 88 if the bisector passes through the center of the circle.

225. Concyclic points. Three or more points which lie on a circle are called *concyclic* points.

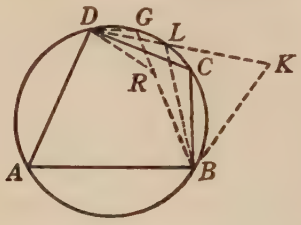
EXERCISES

89. Prove that a circle will pass through the vertices of a right triangle if the center is the mid-point of the hypotenuse and the radius is half the hypotenuse.

90. Prove that the vertices of a rectangle are concyclic points.

Theorem 23

226. *If two opposite angles of a quadrilateral are supplementary, a circle can be circumscribed about it.*



Given the quadrilateral *ABCD*, with the angles *A* and *C* supplementary.

To prove *that a circle can be circumscribed about ABCD.*

Proof

1. Let ABD be a circle passing through A, B, and D. — 1. § 224.

Now the fourth vertex may fall (1) outside the circle, as at K, (2) inside the circle, as at R, or (3) upon the circle, as at C.

(1) Let L be the point where one of the sides of $\angle K$ cuts the circle. Complete the quadrilateral $BADL$.

2. $\angle A + \angle BLD = 180°$. — 2. § 219.
3. $\angle BLD > \angle K$. — 3. § 144.
4. Hence $\angle A + \angle K < 180°$. — 4. From statements 2 and 3.

But this is contrary to the hypothesis. Therefore the fourth vertex cannot fall outside the circle.

(2) Let G be the point where BR cuts the circle. Draw chord GD.

5. Then $\angle A + \angle G = 180°$. — 5. Why?
6. $\angle BRD > \angle G$. — 6. Why?
7. Hence $\angle A + \angle BRD > 180°$. — 7. From statements 5 and 6.

But this contradicts the hypothesis. Hence the fourth vertex cannot fall inside the circle. Then it must fall on the circle, as at C.

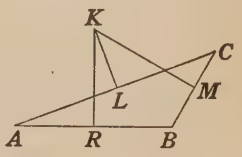
Exercises

91. Prove by Theorem 23 that a circle can be circumscribed about a rectangle.

92. Are the vertices of a square concyclic? of a rhombus? of a parallelogram? Prove.

93. In the trapezoid $ABCD$ the angle A equals the angle B and the angle C equals the angle D. Are the points A, B, C, and D concyclic? Prove.

94. K is a point outside the triangle ABC. KR, KL, and KM are the perpendiculars to AB, CA, and BC respectively, produced if necessary. Prove that the points A, R, L, and K are concyclic, that the points C, L, K, and M are concyclic, and that the points B, K, R, and M are concyclic, and locate the center of each circle.



Miscellaneous Exercises on Circles

95. The chords AB and CD intersect at O. The angle AOC is 30° and the arc DB is 42°. Find the arc AC.

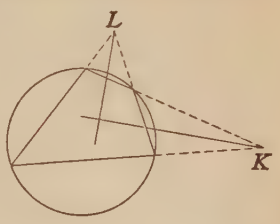
96. The parallel sides of an inscribed trapezoid subtend arcs of 100° and 110° respectively. Find its angles.

97. $ABCD$ is an inscribed quadrilateral. The arc AB is 48° and the arc BC is 92°. The arc DC is one fourth of the arc AD. Find the number of degrees in each angle of the polygon.

98. An inscribed angle A intercepts an arc y, and a central angle B in the same circle intercepts an arc x. If x plus y equals 98° and the angle B is 8° more than three times the angle A, find the angle A and the angle B.

99. KR and ML, two sides of the inscribed quadrilateral $KRLM$, when produced meet in A. The diagonals of the quadrilateral meet at O. From H on AK a tangent HB is drawn touching the circle at B so that the arc BK is three times the arc BR. The arc RL is 70°, the arc ML is 80°, and the arc MK is 130°. Find the number of degrees in the angles BHK, KAM, RLM, KOR, and ARL.

100. Assign numerical values to the angles of an inscribed quadrilateral such that its opposite sides produced will meet in K and L respectively. Then bisect the angle K and the angle L and determine the angle between the bisectors.



101. A chord of a circle equals the radius. Show that the minor arc of the chord is 60°.

102. AB is a diameter, and the chord AK is equal to the radius. Draw BK and prove that the angle A is 60° and that the angle B is 30°.

103. Chords AB and AC are equal to the radius. Draw BC and determine the number of degrees in each angle of the triangle ABC.

104. If the two nonparallel sides of a trapezoid are equal, its vertices are concyclic.

105. AB, a chord of a circle, is the base of an isosceles triangle whose vertex C is without the circle and whose equal sides intersect the circle in D and E respectively. Prove that CD is equal to CE.

106. A is the center of a circle and K is any outside point. R is the mid-point of AK. A circle with R as the center and RA as the radius cuts the first circle in H and L. Prove that KH and KL are tangent to the first circle.

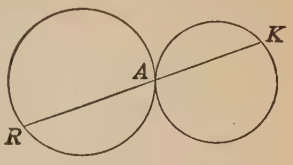
107. The perimeter of an inscribed equilateral triangle is equal to one half the perimeter of the circumscribed equilateral triangle.

108. The diagonals of an inscribed equilateral polygon of five sides are equal.

109. The bisectors of two opposite angles of an inscribed quadrilateral meet the circle in A and B. Prove that AB is a diameter of the circle.

110. ABC is an inscribed equilateral triangle. DE joins the middle points of arcs BC and AC. Prove that DE is trisected by the sides BC and AC.

111. Any secant through the point of contact of two tangent circles subtends arcs of which the two in one circle contain the same number of degrees respectively as the two in the other circle.



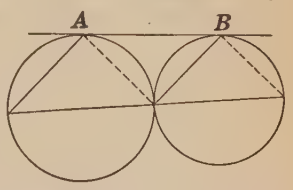
HINT. Draw the common tangent.

112. A secant through the point of contact, A, of two tangent circles cuts them in K and R respectively. Prove that the radii drawn to K and R are parallel.

113. AB, a diameter, is extended half its length to K. Tangents from K cut the tangent at A in R and L respectively. Prove that the triangle KRL is equilateral.

114. ABC is an inscribed triangle. The bisectors of angles A, B, and C meet in D. AD produced meets the circumference in E. Prove that DE is equal to the chord CE.

115. Two circles intersect at A and B, and a tangent to each circle is drawn at A meeting the circles at R and S respectively. Prove that the triangles ABR and ABS are mutually equiangular.



116. A common external tangent is drawn to two circles which are externally tangent. Show that the chords drawn to the points of tangency from the points where the line through the centers cuts the circles are parallel in pairs.

CONSTRUCTIONS

227. Problems of construction. Thus far it has been assumed that such figures as were needed could be drawn with a sufficient degree of accuracy to serve as aids in the demonstration of the exercises and theorems. Methods of *constructing* figures composed of straight lines or of circles or of a combination of the two will now be studied. These methods are theoretically exact. We say theoretically because, however sharp our pencil,

we cannot draw a geometrical line; nor can we draw, even with the best ruler, a line which is perfectly straight. While the figures that result even with exact methods are not absolutely accurate, they are precise enough for many practical purposes; and what is more important from a scientific point of view, they enable us to demonstrate the existence of many geometric relations which until now we have been obliged to assume. For example, in § 42 and other places the existence of a perpendicular from a point to a line was assumed. In § 235 we shall show how to construct such a line.

228. Postulates of construction. The possibility of three simple operations, which underlie all the construction work of elementary geometry, is assumed:

1. *Through any two points a straight line can be drawn.*

2. *A given line-segment can be extended indefinitely beyond either extremity.*

3. *A circle can be described with any given point as a center and any line of a given length as a radius.*

There are, besides the three assumptions here explicitly stated, a very great number of others which are implicitly taken for granted. For example: Every straight line is continuous; Every angle is divisible into any number of equal parts; Two straight lines can intersect in only one point; etc.

229. Instruments used in construction. From the postulates just stated it follows that the instruments which we shall require are only two: the ruler (straightedge) and the compasses.

The ruler is used for drawing a straight line between two points and for extending a given straight line-segment.

The compasses are employed for drawing circles and arcs and are also used for transferring distances; that is, for making one line-segment equal to another one.

The ruler and the compasses are the two instruments which from the earliest times have been associated with the problems of

geometry. We still limit ourselves to their use for several reasons. First, they are easy to obtain and simple to use. Second, although many of the constructions here performed can be carried out more simply by a T-square or a parallel ruler, still the two instruments most essential to a draughtsman are the ruler and the compasses. Third, to include another instrument — for example, one making possible the drawing of ellipses — would so enlarge the field of construction work that even an elementary knowledge of it could not be obtained in the time available.

230. Solution of a construction problem. The solution of a construction problem may best be divided into four parts:

1. **Given.** The length of certain lines, the size of certain angles, the position of certain points or lines, must be definitely stated, for they are the "given" elements.

2. ***Required.*** A precise statement of what is to be done must be made.

3. **Construction.** The figure "required" may well be first sketched in outline but must also be actually constructed by the drawing of straight lines and circles, the whole properly lettered, and the portion of it which is the required figure specified.

4. **Proof.** Here it is necessary to assume that the required figure has been constructed as stated in 3, and to prove that if so constructed it is the required figure.

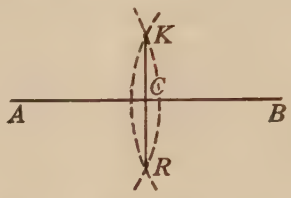
This means that 1 and 3 furnish the hypothesis on which the proof is based.

The written solution of a construction should be accompanied by a figure showing all the arcs whose intersections determine the important points.

The student should note carefully how Constructions 1–9, which follow, illustrate the various points of § 230, and he should observe them in the solutions he himself works out. He should also note that the "given" and the "required" lines in the figures are continuous and all other lines are dotted.

Construction 1

231. *Bisect a given line-segment.*



1. **Given the straight line-segment** AB.

2. ***Required*** *to find a point* C *on* AB *such that* $AC = CB$.

3. **Construction.** Describe two arcs with A and B as centers and with a radius great enough to give two points of intersection, K and R.

Draw KR cutting AB at C.

Then $AC = CB$.

4. **Proof.** $RA = RB$ and $KA = KB$. Const.

Therefore KR is the mid-perpendicular of AB. § 118

NOTE. In performing this construction a radius should be taken large enough to make points R and K fall rather far apart. A line determined by two points very near each other cannot be drawn as accurately as one determined by points a considerable distance apart.

QUERY 1. How can the mid-perpendicular of a line be constructed?

QUERY 2. How can a circle be drawn through two fixed points?

QUERY 3. How can a circle be drawn through three fixed points not in a straight line? (See § 224.)

QUERY 4. In Construction 1 does AB bisect each arc KR? Why?

EXERCISES

117. Divide a line into four equal parts.

118. Construct the mid-perpendicular of a given line.

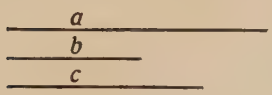
119. Bisect a given arc.

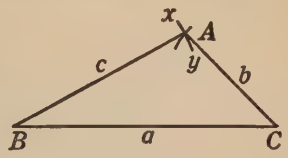
HINT. Construct the mid-perpendicular of the chord of the arc.

120. Circumscribe a circle about a given triangle.

Construction 2

232. *Construct a triangle, given the three sides.*





1. **Given the three lines *a*, *b*, and *c*.**

2. ***Required*** *to construct a triangle with a, b, and c as sides.*

3. **Construction.** Construct the line BC equal to a.

With B as the center and c as the radius, describe the arc xy.

With C as the center and b as the radius, describe an arc cutting arc xy at A.

Draw AB and AC.

Then ABC is the required triangle.

4. **Proof.** $BC = a$, $CA = b$, and $AB = c$. Const.

NOTE. In constructions and computations relating to triangles it is a convenient and a fairly general custom to denote the angles of any triangle ABC by the capital letters A, B, and C, and to denote the sides opposite these angles by the small letters a, b, and c respectively.

233. Altitude of a triangle. An *altitude* of a triangle is a perpendicular from any vertex to the side opposite, produced if necessary.

EXERCISES

121. Construct an equilateral triangle with a given line as a side.

122. Construct an isosceles triangle, given the base and the altitude upon it.

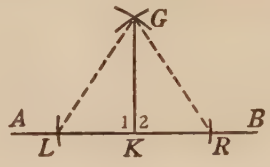
123. Construct a parallelogram, given two adjacent sides and one diagonal.

QUERY 1. How can an angle of 60° be constructed? an angle of 120°?

QUERY 2. How is the supplement of a given angle constructed?

Construction 3

234. *At a given point in a line construct a perpendicular to the line.*



1. **Given the line *AB* and the point *K* on *AB*.**

2. ***Required** to construct a ⊥ to AB at K.*

3. **Construction.** With K as the center and any convenient radius, describe two arcs cutting AB at L and R respectively.

With L and R as centers and a greater radius than before, describe two arcs intersecting at G. Draw line GK.

Then GK is the required perpendicular.

4. **Proof.** Draw GL and GR.

In $\triangle GLK$ and GRK,

	$KL = KR$ and $GL = GR$.	Const.
	$GK = GK$.	Why?
	$\triangle GLK = \triangle GRK$.	Why?
	$\angle 1 = \angle 2$.	Why?
Therefore	GK is ⊥ to AB.	Why?

Exercises

124. Construct a right angle.

125. Given an acute angle, construct its complement.

126. Construct a right triangle, given the two shorter sides.

127. Construct a tangent to a circle at a given point on it.

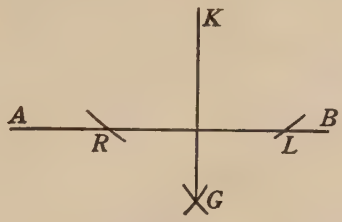
128. Construct a right triangle, given the hypotenuse and a side.

129. Construct a rectangle, given one side and a diagonal.

Query. How can an isosceles right triangle be constructed? an angle of 45°?

Construction 4

235. *From a given point outside a line construct a perpendicular to the line.*



1. **Given the line *AB* and the point *K* outside the line.**

2. ***Required*** *to construct a* $\perp$ *to* AB *from* K.

3. **Construction.** With K as the center and a radius great enough to cut AB, describe arcs cutting AB at points R and L.

Then with R and L as centers and any radius greater than half of RL, describe two arcs intersecting at G.

Then draw KG.

Then KG is the required perpendicular.

4. **Proof.** $KR = KL$ and $GR = GL$. Const.

Therefore KG is perpendicular to RL and AB. § 118

Exercises

130. Given a triangle having one obtuse angle, construct its three altitudes.

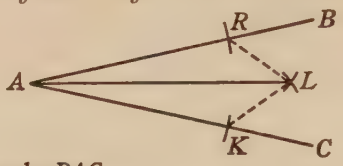
131. Through a given point outside a line construct a parallel to a given line.

Hint. Use Construction 4 and then Construction 3.

Note. In determining points by the intersections of arcs, it is desirable to make the arcs cut each other as in Construction 4. Then the points of intersection can be definitely located. If the arcs cut each other at a small angle, they appear to coincide for some distance near the point of crossing, and the exact point of intersection is not clearly determined.

Construction 5

236. *Bisect a given angle.*



1. **Given the angle *BAC*.**

2. ***Required** to bisect the angle BAC.*

3. **Construction.** With A as the center and any radius AR, describe two arcs cutting the sides of $\angle BAC$ at R and K. Then with any radius greater than half RK and with R and K as centers, describe two arcs intersecting at L.

Draw AL.

Then $\angle BAL = \angle CAL$.

4. **Proof.** Draw RL and KL.

Then $\triangle ARL = \triangle AKL$. Why?

Therefore $\angle BAL = \angle CAL$. Why?

Query 1. May the radius of the arcs which intersect at L equal the radius of arc RK?

Query 2. What point is equidistant from the sides of a triangle?

Query 3. How can such a point be located?

Query 4. How can an angle of 30° be constructed? an angle of 15°?

Query 5. How can an angle of 45° be constructed? of $22\frac{1}{2}$°? of 135°? of 150°?

Query 6. How can a right angle be trisected?

Query 7. Can any given angle be trisected?

Query 8. In Construction 5 does AL bisect the arc KR? Why? Is the chord RK perpendicular to AL? Why?

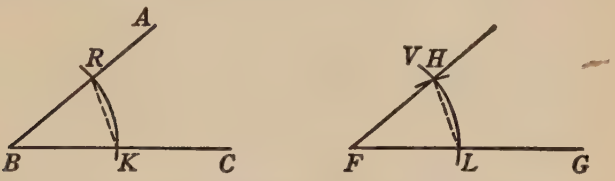
Exercises

132. Inscribe a circle in a given triangle.

133. Given a circular arc AB, find by construction the center of the circle of which the arc is a part.

Construction 6

237. *Given one side and the vertex, construct an angle equal to a given angle.*



1. **Given *FG* one side and *F* the vertex of an angle, and the angle *ABC*.**

2. ***Required*** *to construct an angle equal to $\angle ABC$ with F as the vertex and FG as one side.*

3. **Construction.** With B as the center and any radius, describe an arc cutting AB at R and BC at K. With BK as the radius and F as the center, describe arc VL cutting FG at L. With chord KR as the radius and L as the center, describe an arc cutting arc VL at H. Draw FH.

Then $\angle F = \angle B$.

4. **Proof.** Draw RK and HL.

Then $\triangle BKR = \triangle FLH$. Why?

Therefore $\angle F = \angle B$. Why?

Exercises

134. Construct a triangle, given two sides and their included angle.

135. Construct a triangle, given one side and the two adjacent angles.

136. Construct a triangle, given two of its angles. How many such triangles are possible?

137. Construct a triangle, given two sides and the angle opposite one of them.

138. Given a, b, and angle A. Determine the number of possible triangles if angle A is acute and a is less than b.

139. Proceed as in Ex. 138 if a is equal to b and the angle A is acute.

140. Proceed as in Ex. 138 if a is greater than b and (1) the angle A is acute, (2) the angle A is right, (3) the angle A is obtuse.

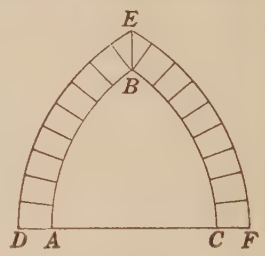
141. Construct an isosceles triangle, given the base and the vertical angle.

142. Through a given point outside a line draw a line parallel to the given one.

HINT. Draw any line through the point cutting the given line at K. Then at the given point construct an angle equal to one of the angles at K so that the two will form a pair of corresponding angles.

143. Carry out the construction of Ex. 142 by using § 235 only.

The adjacent figure gives the outlines of a Gothic arch. The curves used in such arches are circles.



QUERY 1. Are the centers of the arcs BC and EF above or below the line DF?

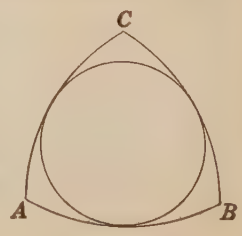
QUERY 2. Is the center of arc BC on AB or AB produced?

QUERY 3. Have the arcs BC and EF the same center?

EXERCISES

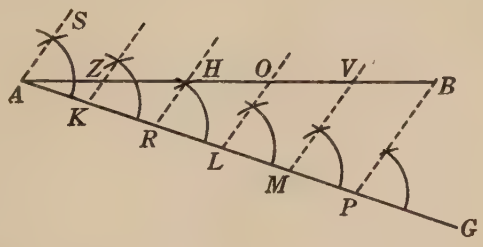
144. Construct a Gothic arch similar to the one above in which AC is two inches and B is one and one-half inches distant from AC.

145. The adjacent form with added detail is often the basis of a design for a window. If the points A, B, and C are the centers of the arcs opposite, show how to construct the inscribed circle.



Construction 7

238. *Divide a given straight line into three or more equal parts.*



1. **Given the line** *AB*.

2. ***Required*** *to divide the line AB into (say) five equal parts.*

3. **Construction.** Draw AG, making $\angle BAG$ about 30°.

Take AK on AG as nearly equal to one fifth of AB as can be estimated and lay off on AG in succession KR, RL, LM, and MP each equal to AK.

Draw BP.

Then at K, R, L, and M respectively construct angles equal to $\angle BPG$ and in a corresponding position.

If necessary, extend the sides of these angles at K, R, L, and M to cut AB in Z, H, O, and V respectively.

Then $$AZ = ZH = HO = OV = VB.$$

4. **Proof.** Draw AS parallel to PB.

By construction, $AK = KR = RL = LM = MP$, and from the construction of the angles at A, K, R, L, and M it follows that AS, KZ, RH, LO, and MV are $\parallel$ to PB. Why?

Therefore AS, KZ, RH, LO, MV, and PB are $\parallel$ to each other. Why?

$$AZ = ZH = HO = OV = VB.$$ Why?

Query 1. Why is it best to make the angle BAG in the above figure acute rather than obtuse?

Query 2. Why is AK made approximately one fifth of AB?

Exercises

146. Construct an equilateral triangle, given a line equal to its perimeter.

147. Construct a square, given a line equal to its perimeter.

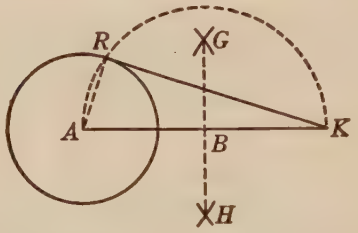
148. Divide a given rectangle into five congruent rectangles.

149. Construct a rhombus, given its perimeter. How many such figures may be constructed with a given perimeter?

150. Construct a triangle, given one side and the two medians from its extremities upon the other sides.

Construction 8

239. *Construct a tangent to a circle from an outside point.*



1. **Given a circle with center A, and a point K outside it.**

2. ***Required*** *to construct a tangent to the circle A from K.*

3. **Construction.** Draw AK. Construct HG, the mid-perpendicular of AK, cutting it at B. With B as the center and BA as the radius, describe a semicircle, AK, cutting the circle A at R. Draw KR.

Then KR is the required tangent.

4. **Proof.** Draw AR. Then $\angle ARK$ is a right angle. Why?

Therefore KR is tangent to the circle A at R. Why?

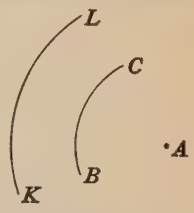
Query. How many tangents may be drawn to a circle from an outside point?

Exercises

151. Given the radii a and b, construct two circles tangent internally and two others externally. (See Exs. 48 and 49.)

152. Given two tangent circles. From what points may two equal tangents, one to each circle, be drawn? Construct two such tangents.

153. A is the center for the arcs BC and KL. Construct a circle between the two arcs tangent to each.

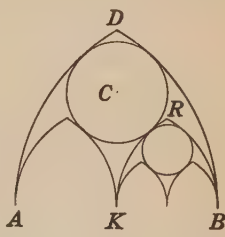


The figure below is the basis of many designs for windows and doors in churches. This design is often employed in the decorations about the entrance and in those about the windows.

The point B is the center from which AD and KR are described.

Query. In what position is the point C with reference to the two arcs KR and AD?

154. If the arches of the adjacent figure are equilateral, that is, if AB equals BD equals AD, construct the design for a given length of AB.



Hint. Observe that one can make the design by starting either with the arc AD or with the shortest arcs of the figure.

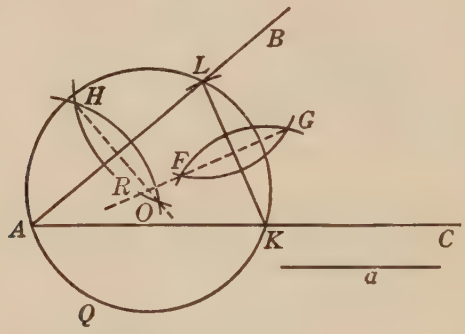
155. Given a circle and a point K outside it, construct a triangle having K as one vertex and its sides tangent to the circle.

156. Given a circle and a point K outside it, construct a rhombus having K as one vertex and its sides tangent to the circle.

157. In Ex. 153, if the arcs BC and KL are on opposite sides of A, construct a circle tangent to each arc.

Construction 9

240. *With a line of given length as a chord construct a segment such that every angle inscribed in it shall equal a given angle.*



1. **Given line *a* and the angle *BAC*.**

2. ***Required*** *to construct a segment whose chord is equal to a and such that every angle inscribed in it shall equal $\angle BAC$.*

3. **Construction.** Select K, a point on AC, such that its distance from AB is less than a. With K as the center and a as the radius, describe an arc cutting AB, as at L. Draw KL. Construct FG and HO, the mid-perpendiculars of KL and AL respectively. Extend HO and GF until they intersect at R. With R as the center and RK as the radius, construct a circle. This circle will pass through A and L. Then the segment LQK is the required segment.

4. **Proof.** $KR = RL$ and $RL = RA$. Why?

Therefore the circle whose center is R and whose radius is RK passes through A and L.

By construction, $KL = a$.

Also, any angle inscribed in segment LQK equals $\angle BAC$. Why?

Exercises

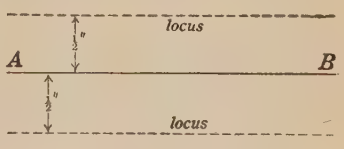
158. Construct a triangle, given one side and the opposite angle. How many such triangles are possible?

159. Construct a triangle, given one angle, the side opposite, and the median on that side.

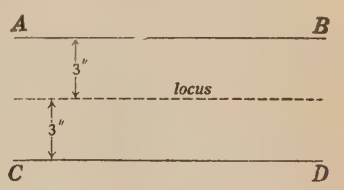
160. Construct a triangle, given one angle, the side opposite, and the altitude on that side.

LOCI

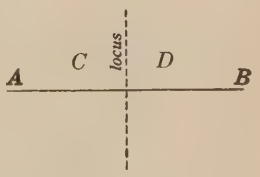
241. Examples of loci. 1. Where are all the points whose distance from a given line AB is half an inch? *Answer:* They are on two lines, one on each side of AB, both parallel to AB and half an inch from it.



2. If two parallel lines AB and CD are six inches apart, where are all the points whose distance from each line is the same? *Answer:* On a line parallel to AB and CD, between them, and three inches from each.

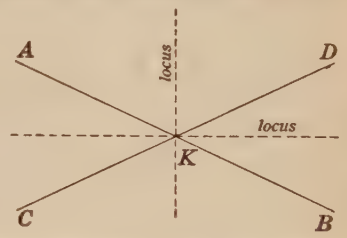


3. Where are all the points whose distance from one end of a given line-segment AB is equal to their distance from the other end? *Answer:* On the line perpendicular to AB at its mid-point. (See §§ 116 and 117.)



Query. Consider any point, as C or D, which is not on the perpendicular bisector of AB. Is it equidistant from A and B?

4. If two lines AB and CD intersect at K, where are all the points equally distant from each line? *Answer:* On the bisectors of the angles formed by these two lines. (See §§ 121 and 122.) The bisectors are perpendicular to each other. Why?



Each of the four preceding questions deals with what is called a *locus* (plural *loci*, pronounced lō′sī), a Latin word meaning "place."

242. Locus. *A locus is a figure containing all the points, and only those points, which fulfill a given requirement.*

For example, the locus determined in 1 and 4 of § 241 consists of two lines. Although every point in either of these lines satisfies the required condition, neither line alone can properly be called *the* locus, because it does not contain *all* the points which fulfill the given requirement.

When looked at in another way:

A locus is the path of a point which moves in such a way that its every position fulfills a given requirement.

In plane geometry only those loci are considered which lie wholly in one plane.

In the following queries determine the locus required. No lengthy detailed proof of the correctness of the result is required, an accurate figure and a brief *common-sense* explanation of why the proposed lines are the required locus is all that is desired. Be sure to determine the entire locus. Sometimes it consists of more than one line.

Query 1. What is the locus of a point two inches from a fixed point A?

Query 2. Is a semicircle the locus of points equidistant from its center? Explain.

Query 3. A fixed line AB, one side of a triangle, is three inches long; the median drawn to it is two inches in length. How many

triangles satisfying these conditions can be drawn? What is the locus of the vertices of all these triangles?

QUERY 4. Given two parallel lines, AB and KR, nine inches apart. What is the locus of a point twice as far from AB as from KR?

QUERY 5. One side of a triangle is two inches and the altitude upon it is one inch. How many triangles satisfy these conditions? What is the locus of the vertices of all these triangles?

QUERY 6. On one side of AB as a hypotenuse how many right triangles can be constructed? What is the locus of all the vertices of the right angles of those triangles? If right triangles be constructed on both sides of AB as the hypotenuse, what is the locus of the vertices of the right angles?

QUERY 7. A fixed point K is the point of intersection of the diagonals of a rectangle. One diagonal is four inches. How many rectangles satisfy these two conditions? What is the locus of the vertices of all the rectangles?

QUERY 8. A man walks across the floor so that he remains equidistant from the east and south walls of a room. What is his path?

QUERY 9. One side of a square is six inches. A point moves so that its distance from one side of the square is always one inch. What is the locus of the point? Is the locus a continuous line?

QUERY 10. Lines are drawn parallel to one side of a triangle and terminating in the other two. What is the locus of their mid-points?

QUERY 11. Lines parallel to one base of a trapezoid terminate in the nonparallel sides. What is the locus of their mid-points?

QUERY 12. Draw KR, any line outside a given square. Inside the square draw fifteen or twenty lines, terminated by its sides and parallel to KR. Locate the mid-points of the lines and draw the line on which these mid-points lie. What is the locus of the mid-points if instead of twenty lines the number is infinite?

QUERY 13. In Query 12 replace the square by an equiangular hexagon and determine the locus.

QUERY 14. A coin, diameter one and one-half inches, is moved around another coin, diameter seven-eighths inches, the two coins constantly touching each other and remaining in the same plane. What is the path of the center of the larger coin?

QUERY 15. K is the center of a fixed circle whose radius is five inches. A circle whose diameter is one inch moves tangent to the first and in its plane. What is the locus of the center of the smaller circle?

QUERY 16. K is any point outside the line AB. Lines are drawn from K and terminate in AB. What is the locus of their mid-points?

QUERY 17. A hay barn, forty feet by eighty, stands in a meadow. A horse is tied outside the barn to one corner by a rope one hundred feet long. Draw a diagram showing the boundary of the area over which he can graze.

QUERY 18. With line AB as a side, triangles are described in which the angle opposite AB is 45°. What is the locus of their vertices?

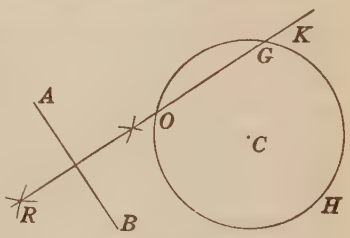
QUERY 19. In Query 18 change 45° to 60° and answer.

QUERY 20. K is any point outside a circle. Draw from K ten or fifteen lines which terminate in the circle but do not cut it. Then locate the mid-points of these lines. What is their locus? If the locus is not apparent, change the position of K, or enlarge the circle, or do both, and repeat the work until the form of the locus is apparent.

QUERY 21. Triangles are constructed on one side of the same base AB, all the angles opposite AB being equal. The bisectors of all the equal angles formed will pass through a single point K. Given AB and the point K, determine the locus of the third vertex of all such triangles.

243. Intersection of loci. Frequently the solution of a construction problem depends on the location of one or more points whose position fulfills two conditions. When this is the case the locus for each condition should be constructed separately. It should be noted that the points of intersection of the loci satisfy both conditions. The determination of their position is often the first and sometimes the most difficult step of the solution.

Example. *Locate all points which are equally distant from two fixed points A and B and half an inch from a third point C.*



Solution. Construct KR, the mid-perpendicular of AB. With C as the center and with a radius of half an inch, describe the circle H. The circle cuts KR in the points O and G, which are the points required in the example.

Proof. KR is the line which contains all the points, and only those points, which are the same distance from A as from B. Why?

The circle C contains all those points, and only those points, whose distance from C is half an inch. Now a point to be the same distance from A and B and a distance of half an inch from C must be on both these loci. O and G are two such points and the only two.

Discussion. The relative position of A, B, and C could be such that the circle CH might cut KR, touch KR, or fail either to cut or touch KR. The first case gives two points, as in the figure, the second would give one, and the third would give none.

Observe that the following problem of construction requires in its solution the location of a point by the intersection of two loci.

EXERCISE **161.** Construct a circle passing through a given point and tangent to a fixed line at a given point of the line.

QUERY 1. A point is three inches from a fixed point A and two inches from another fixed point B. When are there two solutions? one? none? Explain.

QUERY 2. A point is one inch from a line AB and four inches from a point K. May there be no solution for this? two? more than two? Explain.

QUERY 3. A point is equidistant from A and B and six inches from the nearest point of a circle whose radius is two inches. Can there be no solution? more than two? Explain.

QUERY 4. Locate a point which is equidistant from two intersecting lines, and also equidistant from two given points. Can there be no solution? one? more than one? Explain.

QUERY 5. Locate a point which is two inches from a circle whose radius is six inches, and four inches from a given straight line. Discuss the possible solutions.

QUERY 6. AB and AC are two lines. KR is a line which is not parallel to AB or AC. A line HL two inches long and parallel to KR terminates in AB and AC or these lines produced. Locate H and L. Discuss. (See pp. 151–153 for a more exact treatment of loci.)

244. General directions for solving a construction problem. If the solution is not apparent after a brief examination, proceed as follows:

1. Taking a common-sense view of the problem and its data, draw (do not construct) as accurately as possible a figure which approximately satisfies the conditions. Then, using your knowledge of geometry, especially that of loci (§ 243) and of Constructions 1–9 (pp. 132–142), try to devise a way of constructing the required figure. If the desired method is still not apparent, ask yourself questions like the following: Can I construct this part of the figure (a triangle, perhaps) from the given data? If so, can I then construct another part, and finally the whole figure?

2. If no solution results from 1, draw a new figure which conforms to the data but has a different shape from the first. Then repeat the other steps in 1 with the new figure. Sometimes success comes only after several figures have been drawn and studied.

While conforming strictly to the conditions of a construction problem we may select one of many possible positions for the given points and lines and one of many possible sets of values for the lengths for the given distances. The permissible variations here may result in no solution, in one solution, in a definite number of solutions, or in infinitely many solutions. In the following exercises the word *discuss* (see the example of § 243) calls for an analysis of the permissible changes in the data and the consequent variation in the number of solutions.

Miscellaneous Exercises in Construction

162. Construct a parallelogram, given two adjacent sides.

163. Construct a circle of a given radius tangent to a given circle at a given point thereon.

164. Given the three radii, construct three circles each tangent externally to the other two.

Hint. Sketch three circles in the required position and study the distances between their centers.

165. Given two fixed circles, construct a third circle of a given radius tangent internally to the other two.

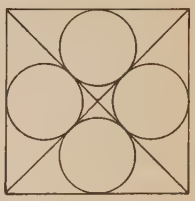
166. Given two fixed circles, construct a third circle of a given radius tangent externally to one and internally to the other. Discuss.

167. Construct a circle having its center in a fixed line and passing through two fixed points.

HINT. What is the locus of the centers of all the circles passing through two fixed points?

168. Construct a circle of a given radius touching a fixed circle and a fixed line.

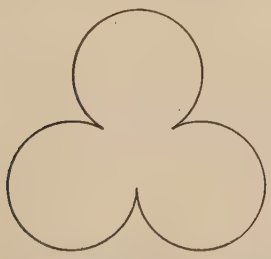
HINT. Where are the centers of all circles of given radius which (1) touch the given line? (2) touch the given circle? Where are the centers of the circles satisfying both conditions?

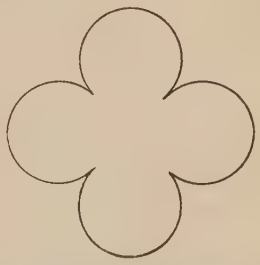


169. Construct four circles inside a square each tangent to one side of the square and to two of the circles.

170. Construct four circles inside a square each tangent to two sides of the square and to two of the circles.

171. Construct the designs given below.





172. Construct a rectangle, given its perimeter and one side.

173. Given two points on a circle as the points of contact of the sides of a circumscribed rhombus, construct the rhombus.

174. Inscribe a regular hexagon in a given circle. (See Ex. 5, p. 95.)

175. Inscribe an equilateral triangle in a given circle.

176. Construct an equilateral triangle with its sides touching a given circle.

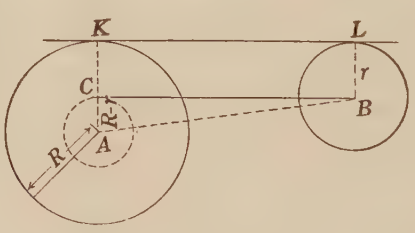
HINTS. Draw an equilateral triangle and inside it draw a circle touching the sides of the triangle at K, R, and L respectively. Draw radii OR, OK, and OL. How many degrees are there in $\angle KOR$? Conclusion?

177. Construct a right triangle, given the hypotenuse and one acute angle.

HINT. What is the locus of the vertex of the right angle of all right triangles constructed on the given hypotenuse?

178. Construct the bisector of the angle between two given nonparallel lines without extending them to their point of intersection.

179. Given two fixed circles, construct their common external tangent.



HINT. Let $AC = R - r$. How is BC drawn? AK? BL?

180. Through one vertex of a triangle draw a line equidistant from the other two vertices. Discuss.

181. Given AB and AC, two intersecting lines, and K any point within the angle BAC. Draw through K a line terminated by AB and AC and bisected by K.

182. Given the triangle ABC. Find by construction one or more points on its circumscribed circle without using its center or its radius.

245. General directions for solving a locus problem. The complete solution of a locus problem which is not at once obvious involves two distinct processes:

1. The drawing of a good figure and locating, as accurately as possible, enough points which answer the requirements to indicate the position and character of the lines composing the desired locus.

Step 1 is really a matter of experiment and discovery, and while it is of the utmost importance, the description of its details is not required in any written or oral solution of a problem.

2. The drawing of the locus discovered in step 1, the assertion that the line or the system of lines so drawn is the locus, and the proof of the assertion. This geometrical proof consists of two parts:

a. The proof that every point on the proposed locus satisfies the given condition.

b. The proof that any point which satisfies the given condition is on the proposed locus.

Frequently step 1 is unnecessary because of the simple character of the locus. As an illustration of step 2 note question 3 of § 241. Then turn to §§ 116 and 117. These two theorems really deal with the locus of a point which is equally distant from the extremities of a line. The first proves that every point of that description is on the mid-perpendicular of the line. This is *a* under step 2. The second proves that every point which is equally distant from the extremities of the line is on the mid-perpendicular of the line. This is *b* under step 2.

Question 4 of § 241 has in a way been touched upon in §§ 121 and 122. These two theorems constitute *a* and *b* respectively under step 2.

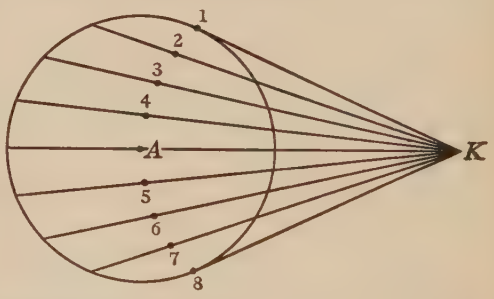
The following example illustrates § 245, and shows how to attack the more difficult locus problems and indicates just what should be stated in the solution itself.

246. Example. Solution of a locus problem.

From point K outside a circle whose center is A secants are drawn. What is the locus of the mid-points of the chords so formed?

Draw a number of secants, each making about the same angle with the two adjacent to it, and locate very accurately the mid-points of the corresponding chords, as in the figure below.

Now it can be seen that points 1, 2, 3, etc., seem to form an arc of a circle. Inspection indicates that the center of this circle is on AK. Closer inspection shows that the mid-perpendicular of the chord from A to point 1 will nearly bisect line AK. It seems very probable, therefore, that AK is a diameter of the circle. It thus appears reasonable to infer that the required locus is that portion of the circle having AK as the diameter which lies within the circle whose center is A.



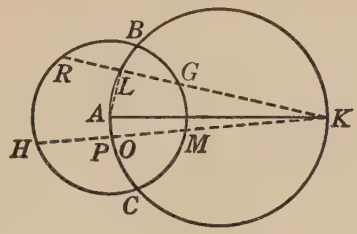
Thus we discover the probable locus. If we had not done so, we ought to have moved K away from or toward the circle, or used a larger circle for A, or both, and to have inspected carefully the resulting figure.

In every case, if we are to discover anything worth while, an accurate figure is absolutely necessary. Moreover, we must not always expect to discover the locus by just one drawing, however carefully it may be done. Finally, however excellent the figure, we must study it attentively, calling to our aid in the process whatever intuition and geometrical imagination we possess.

The preceding work of discovery, while absolutely essential in determining the nature of a locus, need never be written down in the solution of a locus problem. All that need be included in any solution is illustrated by what follows.

Given a circle with center A, and a point K outside the circle.

Required *the locus of the mid-points of all the chords formed by secants from K.*



Solution. On AK as a diameter describe a circle cutting A at B and C. Then arc BAC is the required locus.

Proof. (1) Let L be any point on arc BAC. Draw the secant $KGLR$ and chord AL.

Since AK is a diameter, $\angle ALK = 90°$. Why?

Then $RL = LG$. Why?

Since L is any point on BAC, the preceding proof holds for every point on arc BAC.

(2) Let P be the mid-point of any chord, as HM, formed by the secant KMH. Let HM cut the arc BAC at O. Draw AO.

$\angle AOK = 90°$. Why?

$HO = MO$. Why?

Hence P must coincide with O. Why?

Therefore arc BAC is the required locus.

Exercises

183. Given a point K on a fixed circle, find the locus of the mid-points of all the chords drawn from K.

184. Given a point K within a fixed circle, find the locus of the mid-points of all the chords drawn through K.

185. From a point K outside a fixed circle a number of secants are drawn each cutting the circle in two points and terminating in it. What is the locus of their mid-points and of the mid-points of the external part of each?

Historical Note. In the field of mathematics there is scarcely any name more noted than that of Euclid, the first professor of mathematics in the first university in the world, the University of Alexandria. His great reputation rests securely on his "Elements," a textbook on elementary mathematics (see page 86), devoted mainly to geometry but in part to algebra and to the theory of numbers. The "Elements" summed up in such a masterly way the work of his predecessors that though similar works existed before it, none has come down to us. This last, of course, may have been due in part to the standing which his connection with the University of Alexandria gave him. The known facts regarding Euclid are few. We know that he was a Greek and that he was born, probably at Athens, about 330 B. C. and died about 275 B. C. His selection (300 B. C.) for the University might indicate that he was eminent among the mathematicians of his day. Yet such a conclusion is merely probable. The "Elements" itself offers little evidence on this point, for it is known to be largely a compilation, and whether any important part of it was original with Euclid or whether he made any original discoveries in mathematics is a matter of conjecture.

BOOK III

RATIO, PROPORTION, AND SIMILAR FIGURES

247. Introduction. The results obtained in Book I cluster about the theorems on congruent triangles and those in Book II about circles. In both books equal lines and equal angles were dealt with, but only two magnitudes were considered at a time. It will be found that the work of Book III is largely concerned with figures which have proportional lines and especially with what are termed similar polygons. One property of such polygons is that the lengths of any two sides of one are proportional to the lengths of the corresponding sides of the other. Hence in Book III we shall deal very largely with four magnitudes at a time. As in Books I and II, so also in Book III equal angles will play an important part. But while Books I and II deal continually with *two equal* lines, Book III will be concerned almost wholly with *four proportional* lines. Therefore, some clear notions of ratio, measurement, and proportion, which are really algebraic in character, must be grasped before the strictly geometrical work of Book III can be advantageously taken up.

248. Ratio. The *ratio* of one number, a, to a second number, b, is the quotient obtained by dividing the first by the second, or $\frac{a}{b}$.

The ratio of a to b is also written $a : b$, the colon meaning "divided by."

It follows from the above that all ratios are fractions and all fractions may be regarded as ratios.

Thus $3:2$, $\frac{c}{2x}$, $\frac{a+b}{a-b}$, $\frac{2}{5}$, $6:2$, or $\frac{3}{1}$ are ratios and fractions.

249. The terms of a ratio. The first term, or the numerator, in a ratio is called the *antecedent*; and the second term, or the denominator, is called the *consequent*.

Exercises

Write the following ratios as fractions and, if possible, reduce the fractions to their lowest terms:

1. $4\frac{1}{5} : 2\frac{1}{7}$.
2. 27 inches : 10 yards.
3. $(a^2 - 4) : (a - 2)$.
4. $(a^3 - b^3) : (a - b)$.
5. $\left(4 - \frac{1}{a^2}\right) : \left(2 + \frac{1}{a}\right)$.
6. $\frac{1}{x^2 - 4} : \frac{1}{x^2 - 5x + 6}$.
7. Separate 80 into two parts in the ratio of 2 to 3.
8. Separate 154 into three parts in the ratio of $2:3:6$.
9. The value of a ratio is $\frac{3}{5}$ and the antecedent is 12 less than the consequent. Find the ratio.

250. Measurement. We may speak of the ratio of two concrete numbers if they have a common unit of measure. The ratio of 5 yards to 4 feet is $1\frac{5}{4}$, the common unit of measure being 1 foot. Obviously, no ratio exists between 5 years and 3 feet; that is, a ratio can exist only between magnitudes of the same kind.

If we say a piece of paper contains 54 square inches, we are expressing by the number 54 the ratio of the surface of the paper to the surface of a square whose side is 1 inch.

Every measurement, then, is the determination of a ratio, either exact or approximate.

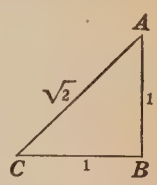
251. Magnitudes which have no common unit of measure. Consider the right triangle ABC in which $AB = BC = 1$.

The student has used in arithmetic the fact, which will be proved in § 284, that

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2.$$

Therefore $\overline{AC}^2 = 1^2 + 1^2,$

or $AC = \sqrt{2}.$



Now $\sqrt{2}$ cannot be exactly expressed by decimals or by a common fraction. Its first seven figures are 1.414213, and the decimal part is nonrepeating. If we suppose AB divided into ten equal parts, one of these would be contained 14 times and a little over in AC. Again, divide AB into 100 equal parts. Then one of them would be contained 141 times and a little over in AC. Again, divide AB into 1000 equal parts. Then one of these would be contained 1414 times and a little over in AC. Here, no matter how minute we make the equal divisions of AB, one of them is never *exactly* contained in AC. This illustrates the fact that AB and AC have no common unit of measure.

The circumference of a circle and its diameter is another example of two magnitudes having no common unit of measure. Here $C \div D = 3.14159 +$ a never-ending, nonrepeating decimal.

252. Commensurable magnitudes. If two magnitudes have a common unit of measure, they are said to be *commensurable.*

253. Incommensurable magnitudes. If two magnitudes of the same kind have no common unit of measure, they are said to be *incommensurable.*

The distinction between commensurable and incommensurable magnitudes is never one which can be observed by our sense of sight or of touch. For instance, there might be two commensurable lines such that no measure greater than one millionth of an inch would be contained exactly in both. Such a unit of measurement is too small to see even with a microscope, so that our senses

would make these lines appear incommensurable; but according to our definition these magnitudes are as truly commensurable as if the common measure were an inch. It appears, therefore, that the distinction between commensurable and incommensurable numbers would never occur to the mechanic, the draftsman, or the engineer, since he could never use his instruments with sufficient accuracy to detect it. Mathematics, however, does not deal with lines of chalk or of ink. Although we use drawings to suggest to our minds the real geometric figures, the geometrical line does not exist outside our thoughts. It is entirely ideal. Hence properties or magnitudes too minute to appeal to our senses, but not beyond the reach of exact thought, may be very real and important in the study of the science of geometry, where one seeks relations which are not merely approximately true but absolutely so.

QUERY 1. The decimal .272727+ is never-ending. Is it incommensurable with 1?

QUERY 2. Reduce $\frac{3}{11}$ to a decimal. What bearing has the result on the answer to the previous query?

254. Proportion. Four numbers, a, b, c, and d, are in proportion if the ratio of the first pair equals the ratio of the second pair.

This proportion is written either in the form (1) $a : b = c : d$ or (2) $\frac{a}{b} = \frac{c}{d}$.

The second form is the usual algebraic way of writing an equation in which each member is a simple fraction. For this reason it is preferable, as the axioms and rules of operations already familiar to the student will seem to him more directly applicable to it than to the first form.

Since a proportion is an equation of a special type, certain special relations exist among the four numbers involved which do not exist in every equation. As the need for them arises, these special relations will be stated as theorems of proportion and proved.

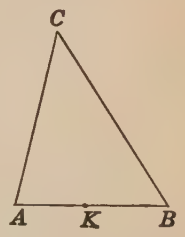
EXERCISES

10. Solve $\frac{4}{5} = \frac{7}{x}$.

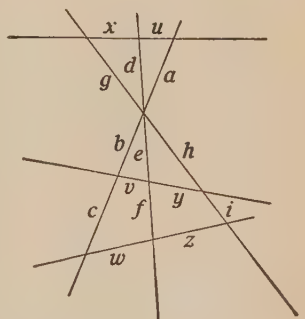
11. Solve $\frac{7}{6} = \frac{3}{x-5}$.

12. Solve $4 : x - 3 = 7 : 2 - x$.

If, referring to the adjacent figure, we write $\frac{AK}{BK} = \frac{AC}{BC}$, we mean $\frac{\text{the length of } AK}{\text{the length of } BK} = \frac{\text{the length of } AC}{\text{the length of } BC}$. Whether the unit of measure is some common standard, as the inch or the centimeter, or a wholly arbitrary length is not stated and really makes no difference. But whatever the unit, AK in the proportion means the *number* of times the unit of length is contained in AK. A like meaning is understood for BK, AC, and BC. Similarly, in all proportions involving the lines of a figure the terms of the proportion are *really numbers*.



255. Corresponding segments of two systems of intersecting lines. If two or more concurrent lines are cut by another group of two or more lines, as in the adjacent figure, certain segments *taken four at a time* are spoken of as *corresponding*. Sometimes *two* segments are spoken of as corresponding to *two* others. The segments of the four following groups in the order named are corresponding:



1. b, c, e, f.
2. v, w, y, z.
3. u, v, x, y.
4. a, g, b, h.

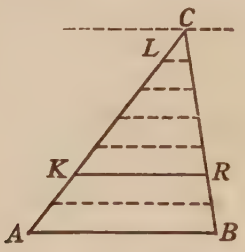
QUERY. Are the following groups corresponding?

1. a, b, g, h. 2. u, w, x, y. 3. a, b, u, v. 4. a, b, h, i.

Usually the term *corresponding* as used in Book III will apply to two or more concurrent lines cut by two or more parallels.

Theorem 1

256. *If a line parallel to one side of a triangle divides a second side into two commensurable segments, it divides the third side into corresponding segments which are proportional.*



Given the triangle *ABC* in which *KR*, parallel to *AB*, intersects *AC* and *BC* at *K* and *R* respectively and forms the commensurable segments *CK* and *KA*.

To prove that $\dfrac{CK}{KA} = \dfrac{CR}{RB}$.

Proof. Apply to AK and KC the common unit of measure CL. Suppose it is contained h times in CK and m times in KA. Through C and the points of division draw parallels to AB, cutting CB.

Then CR will be divided into h equal parts and BR into m equal parts. § 129

Now $\dfrac{CK}{KA} = \dfrac{h}{m}$. (1)

Also $\dfrac{CR}{RB} = \dfrac{h}{m}$. (2) Why?

Therefore from (1) and (2),

$\dfrac{CK}{KA} = \dfrac{CR}{RB}$. Why?

Exercise 13. In the figure above, if AK is 6 inches, KC is 10 inches, and BC is 24 inches, find BR and CR.

257. Alternation theorem of proportion. *If four terms are in proportion, they are in proportion by* ***alternation****; that is, the first is to the third as the second is to the fourth.*

Given $\frac{a}{b} = \frac{c}{d}$. (1)

To prove that $\frac{a}{c} = \frac{b}{d}$. (2)

Proof. Multiplying by bd in (1),

$ad = bc$. (3) Why?

Dividing by cd in (3), $\frac{a}{c} = \frac{b}{d}$. Why?

258. Fourth proportional. A *fourth proportional* to three numbers a, b, and c is the number x if $\frac{a}{b} = \frac{c}{x}$.

Exercises

14. Find a fourth proportional to 7, 3, and 18.

15. Write by alternation $\frac{2}{3} = \frac{4}{6}$.

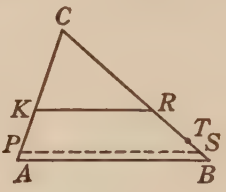
16. Solve the following proportions for x. Write each by alternation and solve. Compare the two results in each case. $4 : 5 = 6 : x$; $a : b = c : x$.

17. Find a fourth proportional to 8, 9, and 36.

Historical Note. The Greeks never succeeded in uniting the ideas of number and magnitude. To them number meant nothing but integers and common fractions. Irrationals were not numbers. In all their arithmetical work they were hampered by a number system even more cumbersome than that of the Romans. This made proportion a rather formidable idea. In modern symbols Euclid's definition of a proportion is as follows: Four numbers, a, b, c, and d, are in proportion if for any integers, m and n, we have $ma > = < nb$, according as $mc > = < nd$.

Theorem 2

259. *If a line parallel to one side of a triangle divides a second side into two incommensurable segments, it divides the third side into two corresponding segments which are proportional to the first pair.*



Given the triangle *ABC* with *KR* parallel to *AB*, and *CK* and *KA* incommensurable.

To prove that $\frac{CK}{KA} = \frac{CR}{RB}$.

Proof. Suppose $\frac{CK}{KA} = \frac{CR}{RB}$ is not true, but that $\frac{CK}{KA} = \frac{CR}{RT}$ is true, and that the point T is between R and B. (1)

Now take a point S between T and B so that RS is commensurable with CR and draw SP parallel to KR.

Then	$\frac{CK}{KP} = \frac{CR}{RS}$.	(2)	§ 256
From (1),	$\frac{CK}{CR} = \frac{KA}{RT}$.	(3)	§ 257
From (2),	$\frac{CK}{CR} = \frac{KP}{RS}$.	(4)	Why?
From (3) and (4),	$\frac{KA}{RT} = \frac{KP}{RS}$.	(5)	Why?

But since $KP < KA$ and $RS > RT$, the numerator of the second member of (5) is less than that of the first, while the denominator of the second member is greater than that of the first. On both grounds

the second member is less than the first and not equal to it. Consequently the assumption (1), which leads to this false result must be incorrect. In like manner we obtain a false result if T is assumed to fall on RB produced. Hence RB must be the true fourth proportional to CK, KA, and CR, and

$$\frac{CK}{KA} = \frac{CR}{RB}.$$

Exercises

18. Take the fraction $\frac{15}{32}$ and try to find an equal fraction having a numerator smaller than 15 and a denominator greater than 32. What point in the proof of Theorem 2 does this illustrate?

19. Prove that part of Theorem 2 in which the point T is assumed to fall on RB produced.

260. *Corollary.* *A line parallel to one side of a triangle and cutting the other two divides them into four corresponding segments which are proportional.*

Outline of proof. Using the figure of § 259, from Theorems 1 and 2 we have

$$\frac{CK}{KA} = \frac{CR}{RB}. \qquad (1)$$

Therefore, from (1), $\dfrac{CK}{CR} = \dfrac{KA}{RB}.$ § 257

Exercises

20. Using the figure of § 259, if CK is 9, KA is 6, and RB is 8, find CB.

21. Using the figure of § 259, if AK is 12, CK is 18, and CB is 25, find CR and RB.

22. Using the figure of § 259, if $AC = 4$, $CR = 24$, and CK is twice KA, find CK, KA and RB.

261. Inversion theorem of proportion. *If four terms are in proportion, they are in proportion by inversion; that is, the second is to the first as the fourth is to the third.*

Given $\frac{a}{b}=\frac{c}{d}$. (1)

To prove that $\frac{b}{a}=\frac{d}{c}$. (2)

Proof. Multiplying in (1) by bd,

$$bc = ad. \tag{3}$$

Dividing in (3) by ac, $\frac{b}{a}=\frac{d}{c}$. Why?

262. Addition theorem of proportion. *If four terms are in proportion, they are in proportion by* ***addition****: that is, the sum of the first two is to either as the sum of the second two is to the corresponding one.*

Given $\frac{a}{b}=\frac{c}{d}$. (1)

To prove that $\frac{a+b}{b}=\frac{c+d}{d}$, (2)

and $\frac{a+b}{a}=\frac{c+d}{c}$. (3)

Proof. Adding 1 to each member of (1),

$$\frac{a}{b}+1=\frac{c}{d}+1. \tag{4}$$

From (4), $\frac{a+b}{b}=\frac{c+d}{d}$. (5)

Write (1) by inversion. Then (3) can be obtained as was (5).

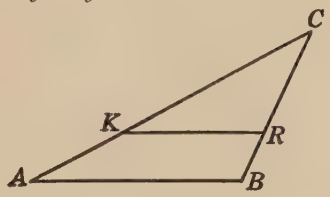
Exercises

23. Write $\frac{4}{9}=\frac{12}{27}$ by inversion and by alternation and each result by addition. Are the statements thus obtained proportions?

24. If $a:b=c:d$, prove that $a+b:a=c+d:c$.

Theorem 3

263. *If a line parallel to one side of a triangle cuts the other two sides, the two sides are in proportion to their corresponding segments.*



Given the triangle *ABC* with *KR*, parallel to *AB*, forming segments *CK* and *KA* of *CA* and *CR* and *RB* of *CB*.

To prove that $\frac{CA}{CB} = \frac{CK}{CR} = \frac{KA}{RB}$.

Proof	$\frac{CK}{KA} = \frac{CR}{RB}$.	(1)	§ 260
From (1),	$\frac{CK + KA}{KA} = \frac{CR + RB}{RB}$.	(2)	§ 262
Thus (2) becomes	$\frac{CA}{KA} = \frac{CB}{RB}$.	(3)	Why ?
From (3),	$\frac{CA}{CB} = \frac{KA}{RB}$.	(4)	§ 257
From (1),	$\frac{CK}{CR} = \frac{KA}{RB}$.	(5)	§ 257

Hence, from (4) and (5), $\frac{CA}{CB} = \frac{KA}{RB} = \frac{CK}{CR}$.

Exercises

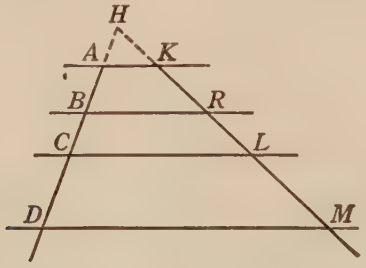
25. In the figure above, if CK is 6, CR is 4, and AK is 10, find BR.

26. In the figure above, if AC is 16, CR is 8, and BR is 6, find AK.

27. In the figure above, if AC is 20, BC is 24, and BR is 6, find CK.

Theorem 4

264. *If two nonparallel lines are cut by three or more parallels, the corresponding segments of the two transversals are proportional.*



Given two nonparallel lines AD and KM cut by the parallels AK, BR, CL, and DM.

To prove *that* $\dfrac{AB}{KR} = \dfrac{BC}{RL} = \dfrac{CD}{LM}$.

Proof. Let AD and KM intersect in H.

Then $\dfrac{HB}{HR} = \dfrac{AB}{KR}$, (1) § 263

and $\dfrac{HB}{HR} = \dfrac{BC}{RL}$. (2) § 260

From (1) and (2), $\dfrac{AB}{KR} = \dfrac{BC}{RL}$. Why?

In like manner it can be proved that $\dfrac{BC}{RL} = \dfrac{CD}{LM}$, and so on for any number of parallels.

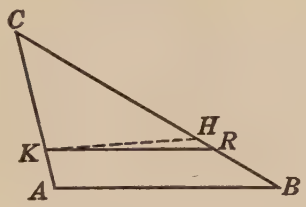
Therefore $\dfrac{AB}{KR} = \dfrac{BC}{RL} = \dfrac{CD}{LM}$.

Query. If the word *nonparallel* is omitted in the statement of Theorem 4, is the resulting theorem true?

Exercise **28.** In the figure above, AB is 8, BC is 7, CD is 10, and KM is 30. Find KR, RL, and LM.

Theorem 5 *(Converse of the Corollary, § 260)*

265. *If a line divides two sides of a triangle into proportional corresponding segments, it is parallel to the third side.*



Given the triangle *ABC* and line *KR* dividing sides *AC* and *BC* so that

$$\frac{CK}{KA} = \frac{CR}{RB}. \qquad (1)$$

To prove that $KR \parallel$ *to* AB.

Proof. Suppose KR is not $\parallel$ to AB. Then draw $KH \parallel$ to AB.

Then $\dfrac{CK}{KA} = \dfrac{CH}{HB}.$ (2) § 260

From (1) and (2), $\dfrac{CR}{RB} = \dfrac{CH}{HB}.$ (3) Why?

From (3), $\dfrac{CR + RB}{RB} = \dfrac{CH + HB}{HB}.$ (4) § 262

Thus (4) becomes $\dfrac{CB}{RB} = \dfrac{CB}{HB}.$ (5)

From (5), $RB = HB.$ Why?

Therefore R and H coincide, Why?

and since KH is $\parallel$ to AB, KR is $\parallel$ to AB.

Exercise **29.** ABC and ABK are two triangles on opposite sides of AB. Through L on BK a parallel to the line from K to C cuts BC in M, and another line through L parallel to KA cuts AB in H. Prove that $BH : BA = BM : BC$.

266. *Corollary.* *(Converse of Theorem 3.) If a line cuts two sides of a triangle so that the two sides are proportional to two of their corresponding segments, the line is parallel to the third side.*

Exercises

30. A line intersects two sides of a triangle and the lengths of the four corresponding segments formed are respectively 10, 6, 18, and 9. Is the line parallel to the third side?

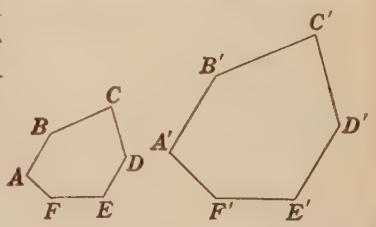
31. Using the figure of § 265, if AK is 10, KC is 15, and CR is 18, what value must BC have so that KR will be parallel to AB?

32. R is a point in the common side of the triangles ABC and ABH. RL, parallel to AC, meets BC in L; and RK, parallel to AH, meets BH in K. Draw LK and CH and prove them parallel.

33. In the triangle ABC a straight line cuts AB at L and AC at R and the third side produced at K. If AR equals AL, prove that $BK : CK = BL : CR$.

Hints. Let CH, parallel to KL, cut AB at H. Then $BK : CK = BL : LH$. Lastly, prove LH equal to CR.

267. Similar polygons. Two polygons are *similar* (symbol $\sim$) if the angles of one are equal respectively to the angles of the other and the sides are proportional each to each.



Thus the polygons $ABCDEF$ and $A'B'C'D'E'F'$ are similar if

(1) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle D = \angle D'$, $\angle E = \angle E'$, and $\angle F = \angle F'$; and

(2) $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EF}{E'F'} = \frac{FA}{F'A'}.$$

268. Corresponding lines. In similar (or congruent) polygons two lines which are in like position with respect to the equal angles are called *corresponding* or *homologous* lines.

In triangles the corresponding sides lie opposite the equal angles.

269. *Corollary.* *Corresponding sides of similar polygons are in proportion by §§ 267 and 268.*

QUERY 1. Must similar polygons have the same number of sides? of angles?

QUERY 2. Are all squares similar? all rectangles? all equilateral triangles?

QUERY 3. Can two polygons be mutually equiangular without being similar? Illustrate.

QUERY 4. Are the sides of the following polygons proportional each to each: (*a*) two squares? (*b*) two rectangles? (*c*) two parallelograms? (*d*) two regular polygons? (*e*) two regular polygons having the same number of sides?

QUERY 5. Can the sides of two polygons be proportional each to each and the two polygons not be similar? Illustrate.

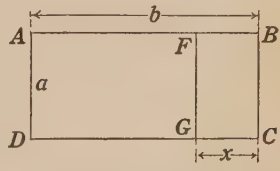
QUERY 6. If two triangles are similar to a third, are they similar to each other? Why?

QUERY 7. Can two rectangles be drawn whose sides are not proportional each to each? Illustrate.

QUERY 8. Can two parallelograms be drawn whose sides are proportional each to each and whose angles are not equal each to each? Illustrate.

QUERY 9. Are there any two polygons which must have their sides proportional each to each if the polygons are mutually equiangular?

QUERY 10. Are there any two polygons which must be mutually equiangular if their sides are proportional each to each?

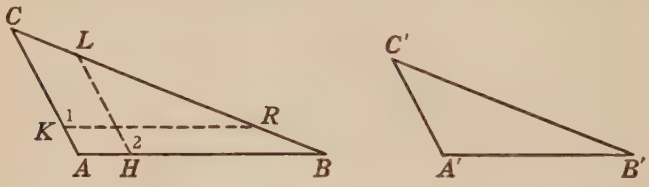


QUERY 11. Are equal triangles similar? Why?

QUERY 12. In the three rectangles of the adjacent figure, x is a third proportional (see § 283) to b and a. Are any two of the three rectangles similar? Which? Prove.

Theorem 6

270. *If two triangles are mutually equiangular, they are similar.*



Given the triangles ABC and $A'B'C'$, in which the angle A equals the angle A', the angle B equals the angle B', and the angle C equals the angle C'.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

Proof. Since $\angle C = \angle C'$, we can place $\triangle A'B'C'$ on $\triangle ABC$ so that point C' falls on point C, side $C'A'$ on CA, side $C'B'$ on CB, and $A'B'$ in the position KR.

By hypothesis,	$\angle 1 = \angle A$.		
Hence	KR is $\parallel$ to AB.		Why?
Therefore	$\dfrac{CK}{CA} = \dfrac{CR}{CB}$.	(1)	Why?
But (1) is the same as	$\dfrac{C'A'}{CA} = \dfrac{C'B'}{CB}$.	(2)	

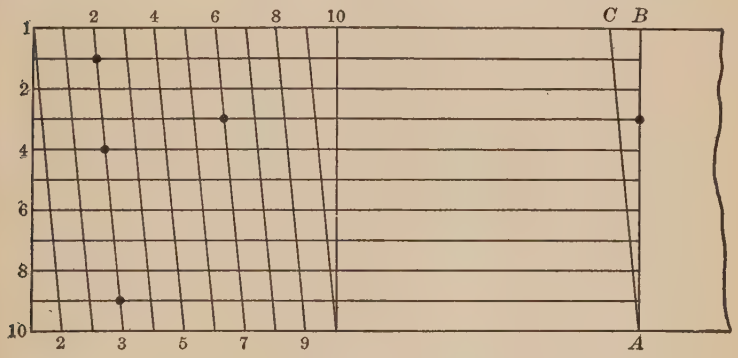
Since $\angle B = \angle B'$, we may now place $\triangle A'B'C'$ on $\triangle ABC$ so that point B' falls on B, side $B'C'$ on BC, side $B'A'$ on BA, and $A'C'$ in the position HL.

By hypothesis,	$\angle A = \angle 2$.		
Hence	HL is $\parallel$ to AC.		Why?
Therefore	$\dfrac{BL}{BC} = \dfrac{BH}{BA}$, or $\dfrac{B'C'}{BC} = \dfrac{A'B'}{AB}$.	(3)	Why?
From (2) and (3),	$\dfrac{C'A'}{CA} = \dfrac{C'B'}{CB} = \dfrac{A'B'}{AB}$.		Why?
Therefore	$\triangle ABC \sim A'B'C'$.		§ 267

271. ***Corollary 1.*** *A line cutting two sides of a triangle and parallel to the third side forms a second triangle similar to the first.*

272. ***Corollary 2.*** *If two angles of one triangle are equal respectively to two angles of another, the triangles are similar.*

273. Gunther's scale. The adjacent figure shows a diagonal scale (enlarged) whose unit is one inch. It is used in connection with a pair of dividers for determining the distance between two points in inches and hundredths of an inch.



To understand the scale consider first the triangle ABC, in which BC represents $\frac{1}{10}$ of an inch. What are the lengths in hundredths of an inch represented by the portions of the lines 1, 2, 4, etc. which are included between lines BA and CA?

Exercises

34. What is the distance from the point marked on AB to the point marked on the oblique line 6–7?

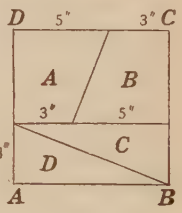
35. What is the distance from each of the three points marked on the oblique line 2–3 to the line AB?

36. In the triangle ABC the side AB is 8, BC is 12, and CA is 16. A line parallel to BC cuts AB at K so that AK is 6 and cuts AC at R. Find the other sides of the triangle AKR.

37. The sides of a trapezoid are 10, 15, 16, and 36 respectively. The two longer sides are parallel. The other two sides are extended to meet. Find the perimeter of the smaller triangle so formed.

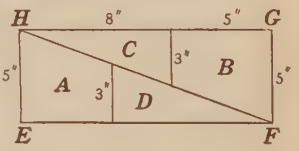
38. In the figure of § 264, if AC is 18, AK is 16, and CL is 24, find HA.

39. A line is drawn from the vertex C of triangle ABC to K on the opposite side, produced if necessary, making angle ACK equal to angle B. What two triangles of the figure are similar? Prove.



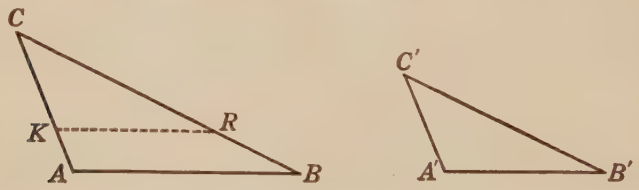
40. CM is a median of the triangle ABC. KRL, parallel to AB, cuts AC in K, CM in R, and BC in L. Prove that KR is equal to RL.

41. If an 8-inch square is cut as indicated in $ABCD$ and the parts arranged in $EFGH$, it may be made to *appear* that the areas of the two figures are equal or that 64 square inches equals 65 square inches. Where is the error? Prove.



Theorem 7

274. *If an angle of one triangle equals an angle of another and the sides including those angles are proportional, the triangles are similar.*



Given the triangles ABC and $A'B'C'$ in which angle C equals angle C' and

$$\frac{CA}{C'A'} = \frac{CB}{C'B'}.$$

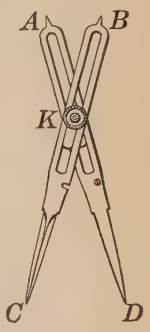
To prove that $\triangle ABC \sim \triangle A'B'C'$.

Proof. Since $\angle C = \angle C'$, we may place $\triangle A'B'C'$ so that point C' falls on C, side $C'A'$ on CA, and side $C'B'$ on CB. Then $A'B'$ will fall in the position KR.

Now, by hypothesis,	$\frac{CA}{C'A'} = \frac{CB}{C'B'}$, or $\frac{CA}{CK} = \frac{CB}{CR}$.	
Hence	KR is $\parallel$ to AB.	§ 266
Therefore	$\triangle KRC \sim \triangle ABC$,	§ 271
or	$\triangle ABC \sim \triangle A'B'C'$.	Why?

Exercises

42. Proportional compasses are often used in enlarging or reducing drawings. These instruments have a clamp, K, which may be set at any point in the slots in the legs. Where must K be placed with respect to A, B, C, and D so that the dimensions of a copy of a drawing will be 60% greater than in the original, AB being the distance between two points in the latter?



43. The bases of a trapezoid are 12 and 18 respectively; its altitude is 10. The nonparallel sides are produced to meet. Find the altitude of each triangle thus formed.

44. The diagonals of quadrilateral $ABCD$ intersect in point K so that $AK : KC = BK : KD$. Prove that $ABCD$ is a trapezoid.

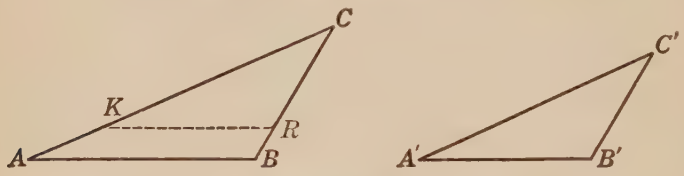
45. Two corresponding medians of similar triangles form two pairs of similar triangles.

46. In the figure of § 274, if CK equals one half AK and AB is 24, find $A'B'$.

47. How would the points C and D of the proportional compasses be placed on the scale of § 273 in order to set the compasses so that the distance CD will be 1.43 inches?

Theorem 8

275. *If the sides of two triangles are respectively proportional, the triangles are similar.*



Given the triangles ABC **and** $A'B'C'$ **in which**

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}. \tag{1}$$

To prove *that* $\triangle ABC \sim \triangle A'B'C'$.

Proof. On CA lay off CK equal to $C'A'$ and on CB lay off CR equal to $C'B'$. Draw KR.

Then $\triangle CKR \sim \triangle CAB.$ § 274

Hence $$\frac{CA}{CK} = \frac{CB}{CR} = \frac{AB}{KR}. \tag{2}$$

But $CK = C'A',$

and $CR = B'C'.$

Hence (2) becomes $$\frac{CA}{C'A'} = \frac{BC}{B'C'} = \frac{AB}{KR}. \tag{3}$$

From (1) and (3), $$\frac{AB}{A'B'} = \frac{AB}{KR}. \tag{4}$$

From (4), $A'B' = KR.$ Why?

Hence $\triangle A'B'C'$ is congruent to $\triangle KRC,$ Why?

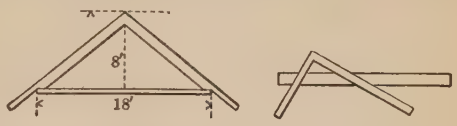
or $\triangle A'B'C' \backsim \triangle ABC.$ Why?

Note. From Theorems 6 and 8 it follows that the triangle is unique among polygons. If either one of the conditions of the definition of § 267 holds for two triangles, the other follows by §§ 270 and 275 as a necessary consequence. But for any polygons except triangles one of the conditions of § 267 may exist without the other.

Exercises

48. A triangle is constructed with the longest sides respectively of three similar unequal triangles. Is the triangle similar to the other three? Prove. Can such a triangle be constructed?

49. A garage is to be 18 feet wide. The roof is to rise 8 feet as shown in the adjacent figure. How shall a carpenter's square be placed on the rafter to mark the proper angle at which to cut the upper end? the lower end?



276. General addition theorem of proportion. *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Given $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}. \tag{1}$$

To prove that $$\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}. \tag{2}$$

Proof. Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=r. \tag{3}$$

Then $$a=br, \tag{4}$$

$$c=dr, \tag{5}$$

$$e=fr. \tag{6}$$

Adding (4), (5), and (6),

$$a+c+e=br+dr+fr=(b+d+f)r. \tag{7}$$

Therefore $$\frac{a+c+e}{b+d+f}=r. \tag{8}$$

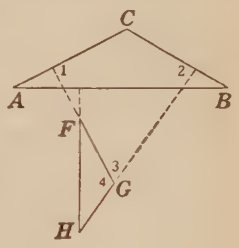
From (3) and (8),

$$\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}.$$

Exercises

50. Prove that the perimeters of two similar triangles are in the same proportion as any two corresponding sides.

51. The sides of a triangle are 9, 10, and 17 respectively. The perimeter of a similar triangle is 108. Find the sides of the latter.



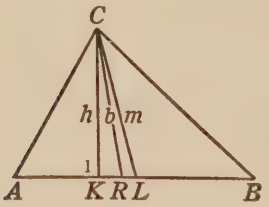
52. Two triangles are similar if their sides are perpendicular each to each.

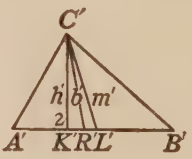
Hint. Extend the perpendicular sides until they meet. What relations exist between the angles 1 and 2? C and 3? 3 and 4? etc.

Theorem 9

277. *In two similar triangles any two homologous sides are proportional to*

(I) *two corresponding altitudes;*
(II) *two corresponding medians;*
(III) *the bisectors of two corresponding angles.*





Given the similar triangles ABC and $A'B'C'$, in which RC bisects the angle ACB and $R'C'$ bisects the angle $A'C'B'$, also CL and $C'L'$, two corresponding medians, and CK and $C'K'$, two corresponding altitudes.

To prove that $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'} = \frac{h}{h'} = \frac{m}{m'} = \frac{b}{b'}. \qquad (1)$$

Proof. (I) In right triangles AKC and $A'K'C'$,

$\angle 1 = \angle 2$ and $\angle A = \angle A'$.

Hence $\triangle AKC \sim \triangle A'K'C'$, § 272

and $\frac{h}{h'} = \frac{AC}{A'C'}$. (2) § 269

By hypothesis, $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$. (3)

From (2) and (3), $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'} = \frac{h}{h'}$. (4)

(II) By hypothesis, $\frac{BC}{B'C'} = \frac{BA}{B'A'} = \frac{\frac{1}{2}BA}{\frac{1}{2}B'A'} = \frac{BL}{B'L'}$,

and $\angle B = \angle B'$.

$\triangle BCL \sim \triangle B'C'L'$. § 274

Therefore $\frac{m}{m'} = \frac{BC}{B'C'}$. (5) Why?

(III) $\triangle BCR \backsim \triangle B'C'R'$. § 272

Hence $\frac{b}{b'} = \frac{BC}{B'C'}$. (6) Why?

From (4), (5), and (6), $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'} = \frac{h}{h'} = \frac{m}{m'} = \frac{b}{b'}$.

278. Corresponding lines of similar polygons. Theorem 9 is a special case of the much more general theorem which follows:

In two similar polygons any two lines similarly drawn are in the same ratio as any two corresponding sides.

Exercises

53. Two triangles are similar if their sides are parallel each to each.

54. Prove that the altitudes of two similar trapezoids are proportional to any two corresponding sides.

55. Prove that two corresponding diagonals of two similar hexagons are proportional to any two corresponding sides.

279. Drawing to scale. Every map is drawn to scale and has (or should have) its scale indicated; that is, a statement should be given saying that one inch (or some other unit of length) on the drawing represents a certain number of inches, feet, or miles, or some other distance, in the country of which the map is a representation. Similarly, the plans of a house, the detailed drawings for the pattern of a small casting, or those for the construction of an ocean liner are all drawn to scale. Such drawings are based on the principles of proportion and the relations of similar triangles. By the use of scale drawings engineers often avoid long calculations and difficult field measurements. Certain lines or angles of a field may be measured and from them an accurate drawing made. Then the measurement of certain distances in the drawing will give the distance between objects in the field, distances which it would frequently be difficult if not impossible to measure or a laborious task to compute by any other method.

The example cited indicates the value of what is called the *graphical method* of solving problems. It is used widely in all kinds of construction work and one of its great advantages is that it presents the results of the whole computation to the eye so that its various relations can be readily grasped. Its greatest advantage for certain classes of problems, however, is its brevity compared to trigonometric or other methods of calculation.

280. Instruments used in graphical constructions. Besides the straightedge and compasses, three other instruments are desirable for graphical solutions. These are a triangle, a protractor, and a scale marked in inches and tenths of inches.

The triangle has one right angle. Usually the other angles are either 60° and 30° respectively, or each is 45°. The right triangle can be used for constructing a perpendicular at the point K of the line AB, as indicated in Fig. 1. A series of parallels can be accurately and quickly drawn by means of the straightedge (or T-square) and the triangle, used in the manner indicated in Fig. 2 on the following page.

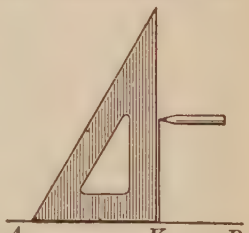


FIG. 1

Fig. 3 shows a protractor, an instrument used for measuring angles and for laying off angles of a required size. The point C is called the center of the protractor. The instrument is placed on the angle ACB in the correct position for measuring it; that is, with the center of the protractor at the vertex and the zero mark on one side of the angle. The other side of the angle coincides with the mark 52°. Hence the angle ACB is equal to 52°. To lay off an angle of 65°, first draw a line CA, then place the protractor in the position indicated and make a point K on the paper opposite the 65° mark. Next remove the protractor and draw CK. Then the angle ACK is equal to 65°.

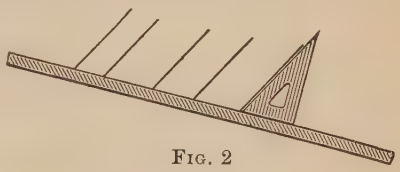


Fig. 2

In the following exercises the student should have a scale marked in inches divided into tenths, a protractor for laying off and measuring angles, a pair of compasses, and a triangle for drawing right angles. The direction "Solve graphically" means "draw to scale the polygons required and draw the lines whose lengths are sought. Then measure these lines and by use of the scale on which the figure has been drawn compute their length."

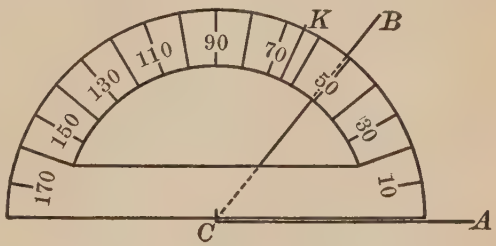


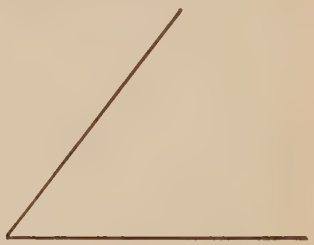
Fig. 3

With a scale marked in inches and tenths of inches, the inches and tenths of an inch in the length of a line can be read directly, and the hundredths of an inch, which are the fractions of a tenth, can be estimated with considerable accuracy.

While graphical solutions are sufficiently accurate for many problems of the engineer, they are not exact. In order to secure good practical results observe the following directions: *Choose as large a scale as possible; use good paper; use a fairly hard drawing pencil*

with a well-sharpened point and draw figures accurately, making all measurements of lines and angles with great care. If still closer approximations are required than can be obtained graphically, it is necessary to use trigonometry.

Exercises for Graphical Solution



56. Measure the number of degrees in the angle of the adjacent figure.

57. Construct an acute angle and measure it.

58. Construct on a given line as a side an angle of 38°.

59. Measure the three lines on the right, obtaining their length to hundredths of an inch. Thus the first line is 1.56 inches in length.

60. Using the protractor and scale, draw an angle of 56° whose sides are 2.3 inches and 3.7 inches respectively.

61. Draw to scale a triangle in which one angle is 70° and the sides including it are 28 inches and 35 inches respectively. Then draw the altitude on the side 35 and determine its length.

62. Draw to scale a triangle whose sides are 15, 20, and 25 respectively.

63. Find the altitude on side 18 of a triangle whose sides are 10, 15, and 18 respectively.

64. Find the median on the longest side of a triangle whose sides are 10, 12, and 18 respectively.

65. Find the length of the bisector of the smallest angle of the triangle whose sides are 9, 12, and 15 respectively.

66. One angle of a parallelogram is 40° and the two sides are 24 and 36 respectively. Find each diagonal.

67. The longest diagonal and the shortest side of a parallelogram are respectively 40 and 16. The angle between them is 28°. Find the other side and the angle it makes with the diagonal.

68. The sides of a triangle are 18, 20, and 34 respectively. Find its three altitudes.

69. The sides of a triangle are 33, 47, and 55. Find the length of the bisector of the greatest angle.

70. Find the shortest median of the triangle whose sides are 115, 252, and 277.

71. Find the least altitude of a triangle whose sides are 164, 225, and 349.

72. If one angle of a rhombus is 60°, show that one diagonal is about 1.73 times as long as the other.

73. In the polygon $ABCDE$ side $AB = 14$, $BC = 12$, $CD = 15$, $DE = 20$, $\angle B = 120°$, $\angle C = 105°$, and $\angle D = 130°$. Draw the polygon to scale and find AE and $\angle A$ and $\angle E$.

74. In the figure of Ex. 73 draw AC and AD and perpendiculars from A to CD, from B to AC, and from E to AD. Then find the length of AC, AD, and each of the three perpendiculars.

75. In the figure of § 290 measure AK, KB, CK, and KD with care. Then substitute in $AK \times KB = CK \times KD$ and find each product. This should give an approximate numerical verification of the truth of Theorem 12.

76. In the figure of § 297 measure AC, BC, KB, and KA. Then substitute in $KB : KA = CB : CA$ and obtain an approximate verification of the truth of this theorem.

77. In a circle of radius 2 inches draw a figure like that of § 294 and measure AB, BC, BK, and BR. Then substitute in $AB \times BC = BK \times BR$ and thus obtain a numerical verification of the truth of the theorem. Is the agreement obtained here closer than that secured in Exs. 75 and 76? Explain.

281. Mean proportional. A mean proportional between two numbers a and b is the number x if $a : x = x : b$.

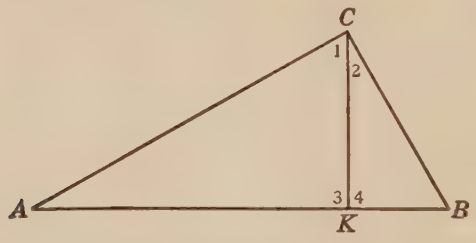
Theorem 10

282. *If a perpendicular is drawn from the vertex of the right angle to the hypotenuse of a right triangle,*

(1) *the two triangles formed are similar to each other and to the given triangle;*

(2) *the perpendicular is a mean proportional between the segments of the hypotenuse; and*

(3) *the square of either side about the right angle equals the product of the whole hypotenuse and the segment adjacent to that side.*



Given CK **perpendicular to hypotenuse** AB **of right triangle** ABC.

(1) ***To prove*** *that* $\triangle ACK \sim \triangle BCK \sim \triangle ABC$.

Proof. $\triangle ACK$, BCK, and ABC are right triangles. Why?

And $\angle A + \angle 1 = \angle A + \angle B$. Why?

Therefore $\angle 1 = \angle B$. Why?

Now $\angle B$ is in $\triangle ABC$ and in $\triangle BCK$, and $\angle 1$ is in $\triangle CAK$.

Therefore $\triangle ACK \sim \triangle BCK \sim \triangle ABC$. § 272

(2) ***To prove*** *that* $AK : CK = CK : KB$.

Proof. By (1), $\triangle ACK \sim \triangle BCK$, and in them $\angle 1 = \angle B$ and $\angle A = \angle 2$.

Then $\dfrac{AK, \text{ opposite } \angle 1}{CK, \text{ opposite } \angle B} = \dfrac{CK, \text{ opposite } \angle A}{KB, \text{ opposite } \angle 2}$. § 269

(3) ***To prove*** *that* $\overline{AC}^2 = AB \times AK$, and $\overline{BC}^2 = AB \times BK$.

Proof. By (1), $\triangle ACK \sim \triangle ABC$, and in them $\angle 1 = \angle B$ and $\angle 3 = \angle ACB$.

Hence $$\frac{AK, \text{ opposite } \angle 1}{AC, \text{ opposite } \angle B} = \frac{AC, \text{ opposite } \angle 3}{AB, \text{ opposite } \angle ACB}.$$ § 269

Then $$\overline{AC}^2 = AB \times AK.$$

In like manner $$\overline{BC}^2 = AB \times BK.$$

283. Third proportional. A third proportional to two numbers a and b is the number x of $a : b = b : x$.

Exercises

78. $ABCD$ is a parallelogram. DR cuts AB in K and CB produced in R. How many similar triangles are thus formed? Prove that $AK : AD = AB : CR$.

79. From any point R on a circle, RK is drawn perpendicular to the diameter AB. Then KH is drawn perpendicular to the radius RL. Show that RH is a third proportional to AL and KR.

80. If the two sides about the right angle of a right triangle are a and $3a$ respectively, the altitude upon the hypotenuse divides it into two segments one of which is nine times the other.

Exs. 81–86 refer to the figure of Theorem 10.

81. If AK is 4 and AB is 20, find CK, AC, and BC.

82. If AC is 6 and AB is 18, find AK, CK, and BC.

83. If AB is 25 and CK is 12, find AK and BC.

84. If AC is 12 and BK is 18, find AK, AB, and BC.

85. If $AC = 10$ and $BK = 8$, find AB.

Hints. $\overline{AC}^2 = AB \times AK$. Let $AK = x$.

Then $$x(x + 8) = 100,$$

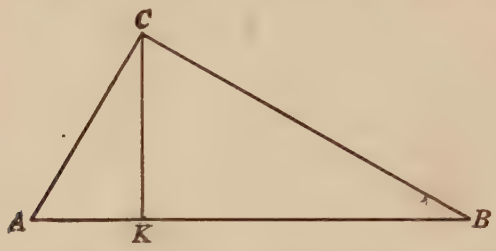
and $$x^2 + 8x + 16 = 116.$$

Whence $$x + 4 = \pm\sqrt{116} = \pm 10.77 \text{ etc.}$$

86. If AK is 12 and BC is 20, find AB.

Theorem 11

284. *In any right triangle the square of the hypotenuse equals the sum of the squares of the other two sides.*



Given the right triangle *ABC* in which *AB* is the hypotenuse.

***To prove* that** $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

Proof. Draw $CK \perp$ to AB.

Then (1) $\overline{AC}^2 = AB \times AK$, by (3) of § 282,

and (2) $\overline{BC}^2 = AB \times BK$.

Adding (1) and (2), $\overline{AC}^2 + \overline{BC}^2 = (AB \times AK) + (AB \times BK)$

$= AB(AK + BK)$

$= AB \times AB$

$= \overline{AB}^2$.

Exercises

Exs. 87–89 refer to the figure of Theorem 11.

87. If AC is 12 and BC is 35, find AB.

88. If AB is 277 and AC is 115, find BC.

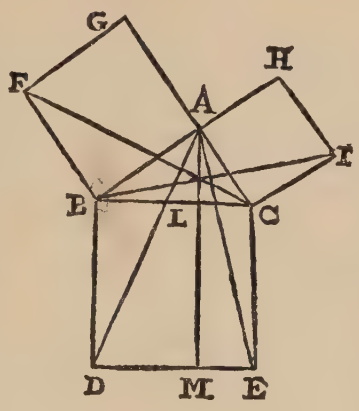
89. If AK is 16 and BC is 15, find BK and AB.

90. Find the diagonal of a rectangle whose sides are respectively 616 and 663.

91. AB and AC are equal sides of the isosceles triangle ABC, AK is the mid-perpendicular of BC, and KR is perpendicular to AC, meeting it in R. Prove that $\overline{AK}^2 = AB \times AR$.

PROP. XLVII.

In right-angled triangles, BAC, *the ſquare* BE, *which is made of the ſide* BC *that ſubtends the right angle* BAC, *is equal to both the ſquares* BG, CH, *which are made of the ſides* AB, AC *containing the right angle.*



Join AE, and AD; and draw AM parallel to CE.

Becauſe the angle DBC *a*=FBA, add the angle ABC common to them both; then is the angle ABD=FBC. Moreover AB *b*= FB, and BD *b*=BC; *c* therefore is the triangle ABD=FBC. But the Pgr. BM *d*= 2 ABD, and the Pgr. *d*BG= 2 FBC (for GAC is one right line by Hypotheſis, and 14. 1.) *e* therefore is the Pgr. BM=BG. By the ſame way of argument is the Pgr. CM=CH. Therefore is the whole BE=BG+CH. *Which was to be demonſtrated.*

a 12. *dx.*
b 29. *def.*
c 4. 1.
d 41. 1.
e 6. *ax.*

Schol.

This moſt excellent and uſeful theoreme hath deſerved the title of *Pythagoras* his theoreme, becauſe he was the inventor of it. By the help of which the addition and ſubſtraction of ſquares are performed; to which purpoſe ſerve the two following problemes.

THE DEMONSTRATION OF THE THEOREM ON THIS PAGE IS FROM A COPY OF BARROW'S EDITION OF EUCLID USED BY A HARVARD STUDENT IN 1735

92. In the figure of Ex. 91 show that $\overline{BC}^2$ is 4 $CR \times AB$.

93. CK is an altitude of the triangle ABC. Prove that $\overline{AC}^2 + \overline{BK}^2 = \overline{AK}^2 + \overline{BC}^2$.

285. Right triangles in which one angle is 45° or 30°. An important application of Theorem 11 is its use in connection with right triangles in which one acute angle is 45° or in which one is 30°. When one angle is 45° the two problems which arise in practice are illustrated in Figs. 1 and 2, while the two problems

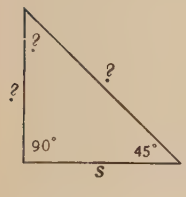


Fig. 1

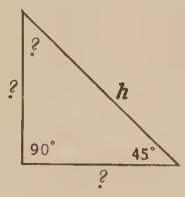


Fig. 2

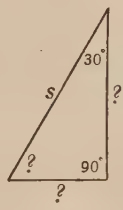


Fig. 3

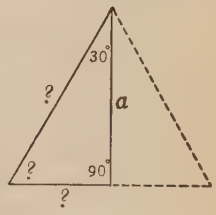


Fig. 4

involving 30° (or 60°) are illustrated in Figs. 3 and 4. Note how the relations proved in Theorems 24 and 23 of Book I are used in 3 and 4 below.

1. Given one angle 45° and one side about the right angle s, to find the other angle and the other two sides.

2. Given one angle 45° and the hypotenuse h, to find the other angle and the other two sides.

3. Given one angle 30° and the hypotenuse s, to find the other angle and the two sides.

4. Given one angle 30° and one side about the right angle a, to find the other angle and the other two sides.

The relation of 3 and 4 above to any equilateral triangle as indicated in Fig. 4 should be carefully noted.

In general, if we know one angle of any right triangle is 30°, 45°, or 60°, and know also any one side, we can compute the other angle and the other sides.

In Exs. 98–102, inclusive, give answers in simplest algebraic form, merely indicating the roots involved.

Exercises

94. If the acute angles of a right triangle are each 45° and one side about the right angle is m, show that the hypotenuse is $m\sqrt{2}$.

95. If the acute angles of a right triangle are each 45° and the hypotenuse is s, show that each leg is $\frac{s}{2}\sqrt{2}$.

96. If one acute angle of a right triangle is 30° and the hypotenuse is m, show that the longer of the other two sides is $\frac{m}{2}\sqrt{3}$.

97. If one acute angle of a right triangle is 30° and the longer side about the right angle is a, show that the hypotenuse is $\frac{2a}{3}\sqrt{3}$.

98. The hypotenuse of an isosceles right triangle is 12. Find the other sides.

99. The side of an equilateral triangle is 13. Find the altitude.

100. The altitude of an equilateral triangle is 7. Find the side.

101. In the triangle ABC, A is 30°, B is 45°, and b is 10. Find a and c.

Hint. Draw a perpendicular to AB from C.

102. In the triangle ABC, A is 30°, b is 12, and c is 16. Find a.

103. In the triangle ABC, A is 45°, B is 120°, and b is 10. Show that c is 2.988+ and a is 8.164+.

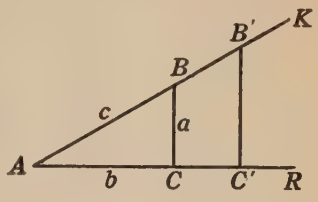
Hint. Draw a perpendicular from C to AB produced.

104. In triangle ABC, A is 45°, b is 20, and c is 40. Show that a is 29.4+.

105. In triangle ABC, A is 30°, B is 135°, and b is 20. Show that c is 7.320+ and a is 14.142+.

286. Introduction to trigonometry. Section 285 treats the important though special problem of computing the parts of a right triangle if one side is given and if it is known that one acute angle is 30°, 45°, or 60°. The general problem, in which one angle has any value between 0° and 90°, requires the methods of trigonometry.

287. Trigonometric ratios. Let KAR be any angle. From any point on either side, as AK, draw BC a perpendicular to the other side. Let $BC=a$, $AC=b$, and $AB=c$.



$$\text{The sine } \angle A = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}},$$

$$\text{cosine } \angle A = \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}},$$

$$\text{and tangent } \angle A = \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}}.$$

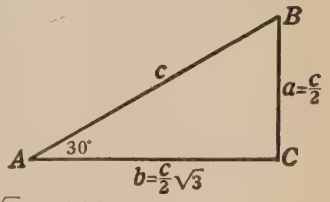
The foregoing ratios are commonly written in an abbreviated form as $\sin A = \frac{a}{c}$, $\cos A = \frac{b}{c}$, and $\tan A = \frac{a}{b}$ respectively.

It is important to observe that the *value of each ratio depends wholly on the size of the angle A*, not on the size of the triangle ABC. Thus

$$\sin A = \frac{CB}{AB} = \frac{C'B'}{AB'}. \qquad \S\ 272$$

These ratios are numbers. Their values to three decimals for every angle from 1° to 89° is given on page 188.

The method of computing the ratios for 30°, 45°, and 60° follows easily from the work on § 285. From the adjacent figure, for example,



$$\sin 30° = \frac{a}{c} = \frac{\frac{c}{2}}{c} = \frac{1}{2} = .5,$$

$$\cos 30° = \frac{b}{c} = \frac{\frac{c}{2}\sqrt{3}}{c} = \frac{1}{2}\sqrt{3} = .866+,$$

and

$$\tan 30° = \frac{a}{b} = \frac{\frac{c}{2}}{\frac{c}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = .577+.$$

Trigonometric Table

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
1°	.017	.9998	.017	45°	.707	.707	1.000
2°	.035	.9994	.035	46°	.719	.695	1.036
3°	.052	.9986	.052	47°	.731	.682	1.072
4°	.070	.9976	.070	48°	.743	.669	1.111
5°	.087	.996	.087	49°	.755	.656	1.150
6°	.105	.995	.105	50°	.766	.643	1.192
7°	.122	.993	.123	51°	.777	.629	1.235
8°	.139	.990	.141	52°	.788	.616	1.280
9°	.156	.988	.158	53°	.799	.602	1.327
10°	.174	.985	.176	54°	.809	.588	1.376
11°	.191	.982	.194	55°	.819	.574	1.428
12°	.208	.978	.213	56°	.829	.559	1.483
13°	.225	.974	.231	57°	.839	.545	1.540
14°	.242	.970	.249	58°	.848	.530	1.600
15°	.259	.966	.268	59°	.857	.515	1.664
16°	.276	.961	.287	60°	.866	.500	1.732
17°	.292	.956	.306	61°	.875	.485	1.804
18°	.309	.951	.325	62°	.883	.469	1.881
19°	.326	.946	.344	63°	.891	.454	1.963
20°	.342	.940	.364	64°	.899	.438	2.050
21°	.358	.934	.384	65°	.906	.423	2.144
22°	.375	.927	.404	66°	.914	.407	2.246
23°	.391	.921	.424	67°	.921	.391	2.356
24°	.407	.914	.445	68°	.927	.375	2.475
25°	.423	.906	.466	69°	.934	.358	2.605
26°	.438	.899	.488	70°	.940	.342	2.747
27°	.454	.891	.510	71°	.946	.326	2.904
28°	.469	.883	.532	72°	.951	.309	3.078
29°	.485	.875	.554	73°	.956	.292	3.271
30°	.500	.866	.577	74°	.961	.276	3.487
31°	.515	.857	.601	75°	.966	.259	3.732
32°	.530	.848	.625	76°	.970	.242	4.011
33°	.545	.839	.649	77°	.974	.225	4.331
34°	.559	.829	.675	78°	.978	.208	4.705
35°	.574	.819	.700	79°	.982	.191	5.145
36°	.588	.809	.727	80°	.985	.174	5.671
37°	.602	.799	.754	81°	.988	.156	6.314
38°	.616	.788	.781	82°	.990	.139	7.115
39°	.629	.777	.810	83°	.993	.122	8.144
40°	.643	.766	.839	84°	.995	.105	9.514
41°	.656	.755	.869	85°	.996	.087	11.43
42°	.669	.743	.900	86°	.9976	.070	14.30
43°	.682	.731	.933	87°	.9986	.052	19.08
44°	.695	.719	.966	88°	.9994	.035	28.64
45°	.707	.707	1.000	89°	.9998	.017	57.29

Exercises

106. Derive the ratios for 45° and for 60° by the use of § 287 and compare the results with the values given in the table on page 188.

107. Construct an angle of 20° with a protractor and complete a triangle like ABC of § 287. Measure a, b, and c, and then compute approximate values for the ratios of 20°. Compare the results obtained with those given in the table.

108. Show that the value of the cosine and the tangent of the angle A do not depend on the size of the right triangle ABC of § 287.

109. Read from the table (*a*) $\sin 30°$, (*b*) $\cos 30°$, (*c*) $\tan 30°$, (*d*) $\sin 18°$, (*e*) $\sin 71°$, (*f*) $\cos 8°$, (*g*) $\cos 32°$, (*h*) $\tan 64°$, (*i*) $\tan 79°$.

110. From the table determine the value of A if

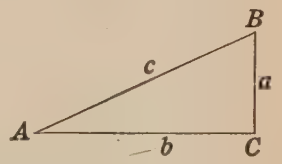
(*a*) $\sin A = .438$ (*d*) $\cos A = .225$ (*g*) $\tan A = 2.904$
(*b*) $\sin A = .731$ (*e*) $\cos A = .956$ (*h*) $\tan A = .325$
(*c*) $\sin A = .966$ (*f*) $\cos A = .629$ (*i*) $\tan A = 1.428$

111. About what value has the angle A if

$\sin A = .590$? $\tan A = .362$? $\cos A = .425$?

288. Use of the trigonometric ratios. If one side and one acute angle of a right triangle can be determined in any way, the other sides can be computed by means of the trigonometric ratios.

Example. In the triangle ABC the angle A is 24° and AC is 42. Find BC.



Solution. $\tan A = \frac{a}{b}$,

or $\tan 24° = \frac{a}{42}$.

$a = 42 \times \tan 24°$

$= 42\,(.445) = 18.69.$

Therefore $BC = 18.69.$

From the triangle ABC of § 287 we have the general results $\sin A = \frac{a}{c}$, therefore $a = c \sin A$ and $c = \frac{a}{\sin A}$; similarly, $\cos A = \frac{b}{c}$, therefore $b = c \cos A$ and $c = \frac{b}{\cos A}$; and $\tan A = \frac{a}{b}$, therefore $a = b \tan A$ and $b = \frac{a}{\tan A}$.

Exercises

It is customary to letter a right triangle as in § 287. With that understanding solve the following right triangles:

112. $A = 25°$, $c = 50$. Find $\angle B$, a, and b.

113. $A = 70°$, $b = 40$. Find $\angle B$, a, and c.

114. $A = 48°$, $a = 30$. Find $\angle B$, b, and c.

115. The shadow of a certain tree on level ground is 60 feet long, and the sun's rays make an angle of 52° with the ground. How high is the tree?

116. A balloon is held captive by a rope 800 feet long. An observer notes that the rope makes an angle of about 76° with the ground, which is level. How high is the balloon?

117. If $c = 125$ and $a = 26$, find $\angle A$, $\angle B$, and b.

118. If $c = 200$ and $b = 106$, find $\angle A$, $\angle B$, and a.

119. If $a = 117$ and $b = 130$, find $\angle A$, $\angle B$, and c.

120. The base of an isosceles triangle is 20 and the angle opposite is 46°. Find the other sides.

121. The sides of a right triangle are 3, 4, and 5. Find to the nearest integer the number of degrees in the acute angles.

122. A chord of a circle whose radius is 10 feet subtends at the center an angle of 40°. How long is the chord?

Hint. Use the cosine of one of the angles to get the half chord.

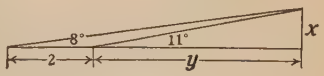
123. The radius of a circle is 32 feet. What angle is subtended at the center by a chord 32 feet long?

124. One of the equal angles of an isosceles triangle is 56°; the base is 18 inches. Find the equal sides.

Hint. Use the cosine of one of the angles.

125. Find the altitude on the base of an isosceles triangle in which the vertical angle is 80° and the base is 32 feet.

126. From one station a balloon was observed to have an angle of elevation of 11°, and at the same instant from another station an angle of 8°. The two stations were at the same level, two kilometers apart, to the south of, and in the same vertical plane with, the balloon. How high was the balloon and how far was it from the nearer station?



HINTS. $\frac{x}{y} = \tan 11°$, or $x = y \tan 11°$.

$\frac{x}{y+2} = \tan 8°$, or $x = y \tan 8° + 2 \tan 8°$, etc.

127. Two observation stations at the same level are seven kilometers apart. An airplane crossing the line connecting the two stations has an elevation of 32° from one station and 40° from the other. Find the height of the airplane at the time.

289. Theorem of proportion on two pairs of equal products. *If the product of two numbers equals the product of two others, the numbers of one pair can be made the first and fourth terms of a proportion of which the numbers of the other pair are the second and third.*

Given $ax = by.$ (1)

To prove that $\frac{a}{b} = \frac{y}{x}.$ (2)

Proof. Dividing (1) by bx, $\frac{a}{b} = \frac{y}{x}.$ (3)

EXERCISES

Write as proportions:

128. $ab = kr.$

129. $xy = 12.$

130. $xy = 1.$

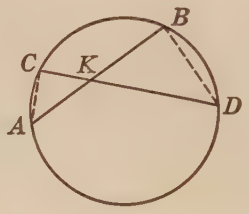
131. $x^2 - y^2 = x^2 - 4x + 4.$

132. $AC \cdot AD = AB \cdot CD.$

133. $\overline{AB}^2 = AK \cdot BK.$

Theorem 12

290. *If two chords of a circle intersect, the product of the segments of one equals the product of the segments of the other.*



Given the chords AB and CD intersecting in point K within the circle.

To prove *that* $AK \times KB = CK \times KD$.

HINTS. Draw AC and BD and prove that $\triangle AKC \sim \triangle BKD$.

Then prove that $AK : CK = KD : KB$, etc.

EXERCISES

134. In the figure above, if AK is 8, BK is 12, and CK is 6, find the length of KD.

135. In the figure above, if AK is 8, BK is 24, and CD is 28, find the length of CK and KD.

136. In the figure above, AB is 24, CK is 8, and DK is 12, find AK and DK to two decimals.

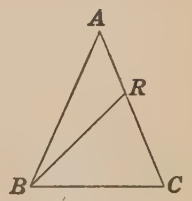
137. On the smooth ice of a lake three vertical standards, each four feet long, were set up in a straight line at intervals of one mile. On looking through a telescope across the top of the first standard to the top of the third it was found that the line of sight fell 8 inches below the top of the second standard. Why? Find from the preceding facts the diameter of the earth.

291. Solution of geometric identities; suggestions for proving the truth of a geometric identity. Theorems 12, 13, and 14 assert that the product of two lines equals the product of two others. With such simple identities one should proceed as follows:

1. Write the identity as a proportion by § 289.
2. Seek two triangles of which the four lines are sides. If the triangles are not complete, try to discover the lines necessary to complete them.
3. Then prove the triangles similar and use § 269. The required result will easily follow.
4. When, as will sometimes happen, the preceding steps fail, try to discover if the four lines of the proportion are corresponding segments of two intersecting lines cutting two parallels.
5. After Theorems 15–18, inclusive, have been mastered, try to use one or more of them if the four preceding steps fail.
6. Consider the possibility and the effect of substituting one line for its equal.

When the identity involves sums and products, an intelligent use of transposition and factoring in connection with some one or more of the preceding steps will frequently result in a solution, as in the following:

Example. In the triangle ABC sides AB and AC are equal. R is a point on AC such that BR equals BC.



Prove that $\overline{BC}^2 - \overline{CR}^2 = AR \cdot CR$.

Transposing, $\overline{BC}^2 = AR \cdot CR + \overline{CR}^2$.

Factoring, $\overline{BC}^2 = CR(AR + CR)$

$= CR \cdot AC$.

Therefore $AC : BC = BC : CR$, etc.

In still more difficult problems one should consider in connection with the preceding, or by itself, the possibility and the effect of substituting the square of one line for the sum or the difference of the squares of two others, or the reverse.

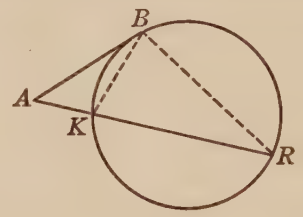
Exercises

138. BC is the common chord of two intersecting circles. The chord BA is tangent to one circle, the chord BK to the other. Prove that $\overline{BC}^2 = AC \cdot CK$.

139. In the parallelogram $ABCD$ a line DE is drawn meeting the diagonal AC in F, the side BC in G, and the side AB produced in E. Prove that $DF \cdot AE = AB \cdot FE$.

Theorem 13

292. *If from a point without a circle a secant terminating in the circle and a tangent be drawn, the square of the tangent equals the whole secant times its external segment.*



Given the secant AR, cutting the circle in K and R, and the tangent AB.

To prove that $\overline{AB}^2 = AR \times AK$.

Hints. Draw BK and BR, and prove that $\triangle AKB \sim \triangle ABR$.

Then prove that $AR : AB = AB : AK$, etc.

293. ***Corollary.*** *If from a point without a circle two secants terminating in it are drawn, the product of one secant and its external segment equals the product of the other secant and its external segment.*

Exercises

140. In the figure of Theorem 13, if AB is 8 and AR is 20, find AK.

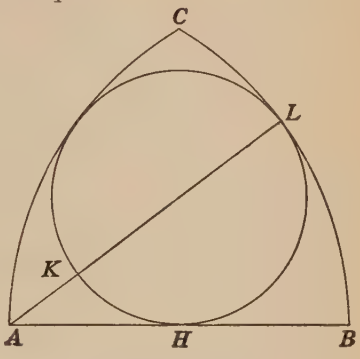
141. A tangent to a circle from A is 12 inches long. AKR is a secant cutting the circle so that the chord KR is 18 inches. Find AK.

142. Tangents to two intersecting circles from any point on their common chord produced are equal.

143. AB and AC are tangents to a circle whose center is K, at B and C respectively. BR drawn $\perp$ to CK produced meets it in R. Prove that $AC : CK = CR : BR$.

144. If the centers of arcs AC and BC are at B and A respectively, the arch is equilateral.

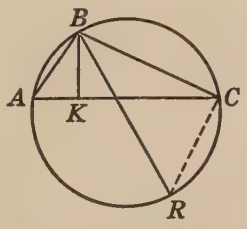
Construct an equilateral arch in which AB is two inches and inscribe a circle in it.



HINTS. Let the radius of the arc $BC = R$. Then $AH = \frac{R}{2}$. Let $KL = 2r$. By § 292, $AK \cdot AL = \overline{AH}^2$. Substituting, $(R - 2r)R = \frac{R^2}{4}$. Whence $R - 2r = \frac{R}{4}$ and $r = \frac{3R}{8}$, etc.

Theorem 14

294. *In any triangle the product of two sides equals the altitude on the third side multiplied by the diameter of the circumscribed circle.*



Given the triangle *ABC*, the altitude *BK*, and *BR* the diameter of the circumscribed circle.

To prove that $AB \times BC = BK \times BR$.

HINTS. Draw CR and prove that $\triangle ABK \backsim \triangle BCR$.

Then prove that $AB : BK = BR : BC$, etc.

Exercises

145. Find the altitude on the side AB of a triangle ABC in which AB is 21, BC is 32, and AC is 20.

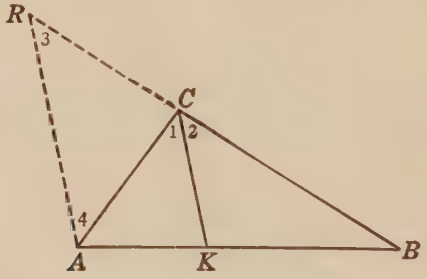
Hints. Draw the altitude CK. Let KA equal x and CK equal y. Then $x^2 + y^2 = 400$ and $(21 - x)^2 + y^2 = 1024$, etc.

146. The sides of a triangle are 4, 13, and 15 respectively. Find the altitude on side 13 and the diameter of the circumscribed circle.

147. The sides of a triangle are 13, 14, and 15 respectively. Find the altitude on the side 14, and the diameter of the circumscribed circle.

Theorem 15

295. *The bisector of an angle of a triangle divides the side opposite into segments proportional to the sides adjacent to them.*



Given the triangle *ABC* in which *CK* bisects the angle *ACB* and meets *AB* at *K*.

To prove that $AK : KB = AC : BC$.

Proof. Through A draw a parallel to CK intersecting BC produced in R.

Then	$AK : KB = RC : CB.$	(1)	§ 260
Now	$\angle 1 = \angle 2.$		Why?
	$\angle 1 = \angle 4.$		Why?

$\angle 2 = \angle 3$. Why?

$\angle 3 = \angle 4$. Why?

$RC = AC$. (2) Why?

From (1) and (2), $AK : KB = AC : BC$. Why?

QUERY. Is Theorem 15 true when AC is equal to CB?

EXERCISES

148. The sides of a triangle are 8, 12, and 15 respectively. Find the segments of the side 15 made by the bisector of the angle opposite.

149. The sides of a triangle are 4, 8, and 10 respectively. The perimeter of a similar triangle is 132. Find the segments of the greatest side of the latter made by the bisector of the angle opposite.

150. CK is the median of the triangle ABC. The bisectors of the two angles at K meet AC and BC in L and R respectively. Prove that LR is parallel to AB.

151. State and prove the converse of Theorem 15.

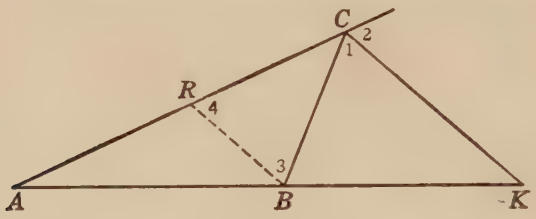
296. Segments of a line made by a point. If in the adjacent figure AB is a line of given length and K is a point on AB or AB produced, KA and KB are called the segments of the line AB.

When the point of division is on the line, the line is divided internally; when the point of division is on the line produced in either direction, the line is divided externally.

Note that in either case the segments are measured from the point of division to the extremities of the line, and that when the point is on the line produced, one segment is longer than the line itself.

Theorem 16

297. *If a triangle is not isosceles, the bisector of an exterior angle at any vertex divides the side opposite externally into segments proportional to the adjacent sides.*



Given the triangle *ABC* in which *CK*, the bisector of the exterior angle at *C*, meets the side *AB* produced at *K*.

To prove that $KB : KA = CB : CA$.

HINTS. Draw BR parallel to CK, cutting AC in R. Is $KB : KA$ equal to $CR : CA$? Why? Compare CB and CR.

QUERY 1. Does the proof of Theorem 16 hold for an isosceles triangle?

QUERY 2. Is Theorem 16 true when the triangle is isosceles?

QUERY 3. What condition determines whether the bisector of the exterior angle at C (using the figure of § 297) meets the side opposite produced to the right or meets it produced to the left?

EXERCISES

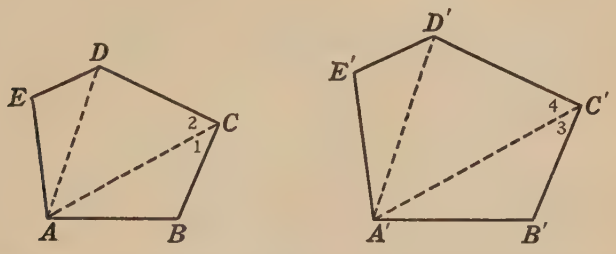
152. The sides of a triangle are 8, 12, and 16 respectively. Find the segments of side 16 made by the bisector of the exterior angle at the vertex opposite.

153. State and prove the converse of Theorem 16.

154. The bisectors of the interior and the exterior angle at C of the triangle ABC cut the base in K and R respectively. Prove that $KA \times RB = RA \times KB$.

Theorem 17

298. *If two convex polygons are similar, they can be divided into the same number of triangles which are similar each to each.*



Given the convex polygon *ABCDE* similar to the convex polygon *A'B'C'D'E'*.

***To prove** that these polygons may be divided into the same number of triangles similar each to each and similarly placed.*

Proof. Draw from two corresponding vertices A and A' all the diagonals possible.

Then	$\frac{AB}{A'B'} = \frac{BC}{B'C'}$ and $\angle B = \angle B'$.		§ 267
Therefore	$\triangle ABC \sim \triangle A'B'C'$.		§ 274
And	$\angle 1 = \angle 3$.		Why?
	$\angle 2 = \angle 4$.	(1)	Why?
	$\frac{BC}{B'C'} = \frac{AC}{A'C'}$.	(2)	Why?
	$\frac{BC}{B'C'} = \frac{CD}{C'D'}$.	(3)	Why?
From (2) and (3),	$\frac{AC}{A'C'} = \frac{CD}{C'D'}$.	(4)	Why?
By (1) and (4),	$\triangle ACD \sim \triangle A'C'D'$.		Why?

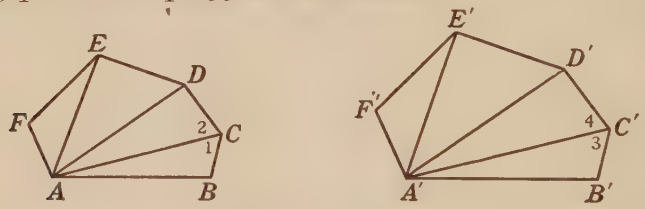
In like manner any triangle in the first polygon can be proved similar to the similarly placed triangle in the second.

QUERY. If the perimeters of two polygons are in the same ratio as the sides taken in pairs, are the polygons similar?

EXERCISE 155. $ABCDE$ and $A'B'C'D'E'$ are similar pentagons. K is a point within the first and K' a point within the second. Lines are drawn from K to the vertices of the first polygon, and from K' to the vertices of the second. If the triangle KAB is similar to the triangle $K'A'B'$, prove that the triangle KBC is similar to the triangle $K'B'C'$.

Theorem 18 *(Converse of Theorem 17)*

299. *If two convex polygons can be divided into the same number of triangles similar each to each and similarly placed, the polygons are similar.*



Given the convex polygons *ABCDEF* and *A'B'C'D'E'F'* divided into triangles so that the triangle *ABC* is similar to the triangle *A'B'C'*, the triangle *ACD* is similar to the triangle *A'C'D'*, etc., and in the same relative position.

To prove *that*

the polygon $ABCDEF \sim$ the polygon $A'B'C'D'E'F'$.

HINTS.

$\angle B = \angle B'$. Why?

$\angle BCD = \angle B'C'D'$, etc. Why?

$\dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$. Why?

$\dfrac{CD}{C'D'} = \dfrac{AC}{A'C'}$. Why?

Therefore $\dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$, etc.

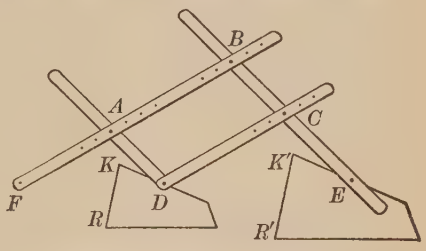
Exercise **156.** Can a line parallel to the bases of a trapezoid divide it into two similar trapezoids? Prove.

Query 1. Are Theorems 17 and 18 true for reëntrant polygons?

Query 2. Why is the word *convex* necessary in the statement of Theorems 17 and 18?

300. The pantograph. The *pantograph* is an instrument for enlarging or reducing drawings, being especially serviceable in tracing curved or broken irregular lines. The instrument is easily understood from the adjacent figure.

The points A and B move in circles about the fixed pivot F, while the points C and D can move in circles about B and A respectively. For these reasons any of the points C, D, or E can, within certain limits, be made to move about at will.



The instrument is adjusted for use by means of the pins at A, B, C, and D. When the pins are properly placed they must be so situated that the figure $ABCD$ will be a parallelogram. If E is a point such that $\triangle FAD \sim \triangle FBE$, the points F, D, and E will remain in a straight line. Then if D is a tracer which follows the drawing, a pencil at E will draw the enlargement. By placing the tracer at E and the pencil at D, a reduced drawing KDR of $K'ER'$ can be made.

Exercises

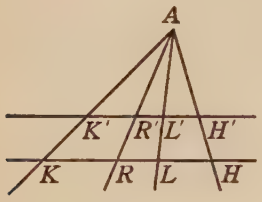
In the preceding figure

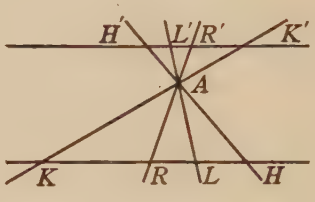
157. Prove that $KR : K'R' = FA : FB$.

158. Determine the boundaries of the area over which D can move.

Theorem 19

301. *If two parallels are cut by three or more concurrent transversals, the corresponding segments of the parallels are proportional.*





Given two parallel lines, KH and $K'H'$, cut by four concurrent lines through A, in K, R, L, H, and K', R', L', and H' respectively.

To prove that $$\frac{KR}{K'R'} = \frac{RL}{R'L'} = \frac{LH}{L'H'}.$$

Proof. Triangles AKR and ARL are similar respectively to triangles $A'K'R'$ and $A'R'L'$. Why?

$$\frac{AR}{AR'} = \frac{KR}{K'R'}. \quad (1)$$ Why?

$$\frac{AR}{AR'} = \frac{RL}{R'L'}. \quad (2)$$ Why?

Therefore $$\frac{KR}{K'R'} = \frac{RL}{R'L'}. \quad (3)$$ Why?

In like manner $$\frac{RL}{R'L'} = \frac{LH}{L'H'}. \quad (4)$$

From (3) and (4), $$\frac{KR}{K'R'} = \frac{RL}{R'L'} = \frac{LH}{L'H'}.$$

QUERY. If KRL and H in the left-hand figure above are fixed and A moves off to a great distance, what does the figure $KRR'K'$ ultimately become?

Exercises

159. Two lines from the vertex of a triangle divide a line which is parallel to the base and which terminates in the other two sides into three parts each equal to one ninth of the base. Prove that the two lines are divided into segments in the ratio of two to one.

160. If n lines from the vertex of a triangle divide a line which is parallel to the base and which terminates in the other two sides into $n+1$ parts each equal to one twentieth of the base, what is the ratio of the two segments into which each of the n lines is divided?

302. The subtraction theorem of proportion. *If four numbers are in proportion, they are in proportion by subtraction; that is, the first minus the second is to either as the third minus the fourth is to the corresponding one.*

Given $\frac{a}{b}=\frac{c}{d}$. (1)

To prove that $\frac{a-b}{b}=\frac{c-d}{d}$, (2)

and $\frac{a-b}{a}=\frac{c-d}{c}$. (3)

Proof. Subtracting 1 in (1),

$\frac{a}{b}-1=\frac{c}{d}-1$. (4) Why?

Whence $\frac{a-b}{b}=\frac{c-d}{d}$. (5)

Write (1) by inversion. Then (3) can be obtained as was (5).

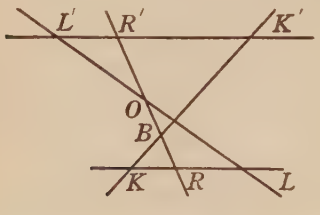
Exercises

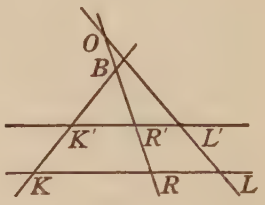
161. Give the proof of (3) as suggested above.

162. If the sides of a trapezoid taken in order are proportional to the sides of another in the same order, are the trapezoids similar? Prove.

Theorem 20 *(Converse of Theorem 19)*

303. *If three nonparallel transversals intercept proportional corresponding segments on two parallels, the transversals are concurrent.*





Given the nonparallel transversals KK', RR', and LL' cutting the two parallels KL and $K'L'$ at K, R, L, and K', R', L', respectively in such manner that

$$\frac{KR}{K'R'} = \frac{RL}{R'L'}. \qquad (1)$$

To prove *that KK', RR', and LL' are concurrent.*

Proof. Let us suppose that KK' and RR' intersect at B, and that RR' and LL' intersect at O. Consider the first figure.

We have $\dfrac{KR}{K'R'} = \dfrac{BR}{BR'}$, (2) § 271

and $\dfrac{RL}{R'L'} = \dfrac{OR}{OR'}$. (3) Why?

From (1), (2), and (3), $\dfrac{OR}{OR'} = \dfrac{BR}{BR'}$. (4)

Then from (4), $\dfrac{OR + OR'}{OR'} = \dfrac{BR + BR'}{BR'}$, (5) § 262

or $\dfrac{RR'}{OR'} = \dfrac{RR'}{BR'}$. (6)

From (6), $OR' = BR'$, hence O and B coincide and the transversals KK', RR', and LL' are concurrent.

The proof for the second figure is similar.

Exercises

163. Prove that the two nonparallel sides and the line joining the mid-points of the bases of a trapezoid are concurrent.

164. Examine the proof of Theorem 20 and determine what changes are necessary to make it apply to the second figure.

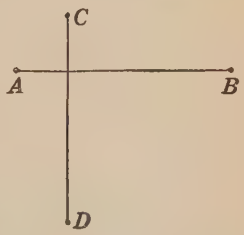
165. If four nonparallel transversals intercept proportional corresponding segments on two parallels, the transversals are concurrent.

Field Problems with a Tape Line

In Exs. 166–173 the use of a fifty-foot tape line, a ball of cord, and a sufficient number of long, straight stakes for marking all important points is assumed. With this equipment three simple field problems are possible:

1. Setting three stakes in a straight line at random distances or at given distances apart.
2. Setting four stakes so that the line (cord) joining two of them is perpendicular to the line (cord) joining the other two (Ex. 168).
3. Setting four stakes so that the line joining two of them is parallel to the line joining the other two (Ex. 169).

166. The height of a flagpole cannot be measured directly, but the length of its shadow can. What simple method with accompanying measurements will enable one to compute the height of the pole?

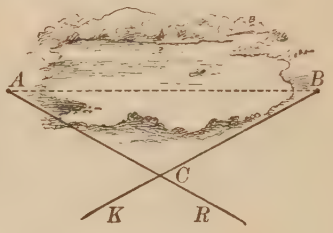


167. Is a triangle whose sides are 3, 4, and 5 respectively or any multiple of these numbers a right triangle? Prove.

168. How can a tape line and the fact just proved be used to locate stakes C and D so that the string stretched between them will be perpendicular to the string stretched from A to B? Could a mason mark the outside line for a foundation in this way? Is it done so in practice? Why not use a square?

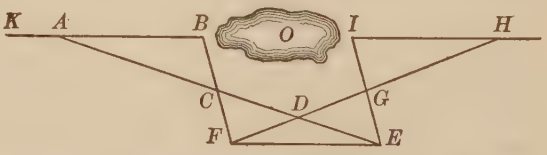
169. How can a tape line be used to place four stakes so that they will determine two parallel lines?

170. In the left-hand figure below, the distance between A and B, two points on opposite sides of a river, is desired but cannot be measured directly. By using a tape what lines in the neighborhood of A can be laid out and measured so that the distance AB can then be computed? Assume values for the measured lines and compute AB.



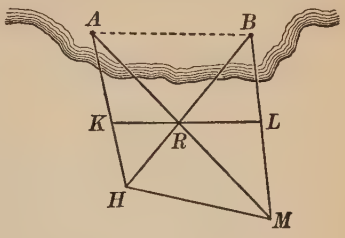
171. In the right-hand figure above, the points A and B are accessible, but the line AB cannot be directly measured. What lines can be laid out and measured that will enable one to compute AB? Assume values for the measured lines and compute AB.

172. In surveying practice a line is prolonged through an obstacle a given distance by the following method: In the adjacent figure suppose it is required to prolong KB through O so



that BH shall be 90 feet. Lay off $AB = 45$ feet. Lay out AE, making an acute angle with AB. Locate C and D so that $AE = 2\,AC = 4\,CD$. Lay out BCF so that $BF = 2\,BC$. Lay out $FDGH$ so that $FH = 4\,FD = 2\,FG$. Lay out EGI so that $EI = 2\,EG$. Then IH will satisfy the conditions set. Prove that this is true. The

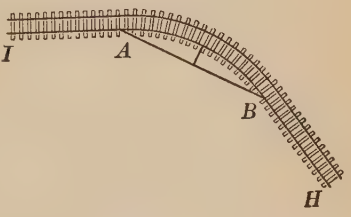
surveyor checks the accuracy of his work by measuring IH. If it equals AB, he considers his work is correct. Prove that this checks the work.



173. In the adjacent figure the distance AB between two inaccessible points offshore is desired. How must the points K, R, L, M, and H (and other points if necessary) be located, and what lines must be measured in order to compute AB? Assume numerical values for the measured lines and compute AB.

Miscellaneous Numerical Exercises

174. Two straight stretches of railroad track, IA and HB, are connected by a circular curve. The chord AB is 100 feet. The distance from the center of the chord to the nearest point of the track is 8 feet. Find the radius of the curve of the inner rail.



175. In triangle ABC, b is 8, a is 12, and c is 10. Find the segments of a made by the bisector of $\angle A$.

176. In the triangle of Ex. 175 find the segments of a made by the bisector of the exterior angle at A.

177. CK is the altitude on the hypotenuse AB of the triangle ABC. AB is 25 and CK is 12. Find AK and AC.

178. In the triangle of Ex. 177, if AB is 13 and AK is 4, find AC and CK.

179. In the triangle ABC, c is 18, b is 20, and a is 24. Find the altitude on side 18.

180. For the triangle of Ex. 179 find the diameter of the circumscribed circle.

181. The altitude of an equilateral triangle is 8. Find to three decimals the length of one side.

182. ABC and AKR are secants to the same circle. AB is 12, chord BC is 8, and chord KR is 15. Find AK and the tangent to the circle from A.

183. The radii of two circles are 8 and 20 respectively. The distance between their centers is 40. How far from the center of the smaller circle does (a) the external common tangent cut the line of centers? (b) the internal common tangent?

184. AB and KR are two chords of a circle intersecting at O. AB is 24, OR is 18, and OK is 3. Find AO and OB.

185. The hypotenuse of a right triangle is 37 and another side is 35. The perimeter of a similar triangle is 308. Find the shortest side of the second triangle.

186. In the triangle ABC, c is 8 inches, $\angle A$ is 45°, and $\angle B$ is 60°. Find b and a.

187. In the triangle ABC, c is 10, b is 16, and $\angle A$ is 30°. Find a.

188. How far is a swimmer from a cliff 400 feet high if when his eye is level with the surface of the water the top of the cliff is just visible? (Use 3960 miles for the earth's radius.)

HINT. Let h = the height of the mountain and d the distance required, both in miles. Then $d^2 = h(h + 2R)$.

189. How high is a mountain if a swimmer thirty miles away can see its top from the surface of the ocean?

HINT. $h(h + 2R) = d^2$. Since h is small compared to $2R$, this becomes very closely $2Rh = d^2$. Whence $h = \frac{d^2}{2R}$.

190. If h is one's height in *feet* above the surface of the ocean and d the distance in *miles* to an object on its surface just visible on the edge of the horizon, show that $d = \sqrt{\frac{3h}{2}}$, approximately.

191. A lookout 100 feet above the surface of the ocean observes an object close to the surface as it comes into view on the edge of the horizon. How far away is the object?

192. K is any point on the semicircle AB. AK cuts the tangent at B in R and BK cuts the tangent at A in L. Prove that AB is a mean proportional between AL and BR.

193. From the vertex D of the parallelogram $ABCD$ a straight line is drawn cutting AB at R and CB produced at K. Prove that CK is a fourth proportional to AR, AD, and AB.

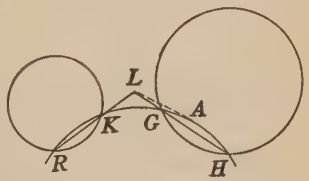
194. AB is a diameter of a circle and KB is a tangent. AK cuts the circle in R. Prove that $AR \cdot AK = \overline{AB}^2$.

195. Two chords of a circle, AB and CD, are produced to meet in K. Then KR is drawn parallel to AD, meeting CB produced at R. Prove that KR is a mean proportional between BR and CR.

196. If two lines parallel respectively to two sides of a triangle are drawn through the point of intersection of the medians, they trisect the third side.

197. Prove Theorem 20 thus: Draw a line through B and L cutting $K'R'$ at H. Then show by the use of the theorems on proportion that L' coincides with H; that is, that the line through L and L' passes through B.

198. Through K, any point in the common base of the triangles ABC and ABR, lines are drawn parallel to AC and AR, meeting BC and BR in the points F and G respectively. Prove that FG is parallel to CR.

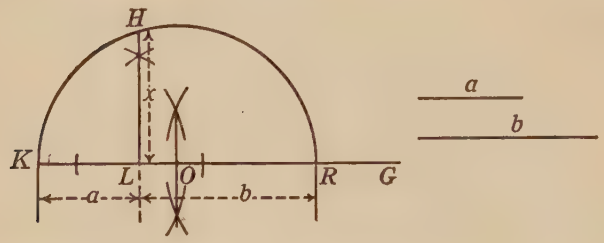


199. Two nonintersecting circles are cut by a third circle. The two common chords intersect at L. Secant LKR cuts the first circle in K and R, and secant LGH cuts the second in G and H. Prove that $LK \times LR = LG \times LH$.

HINT. Draw the tangent LA.

Construction 1

304. *Construct a mean proportional between two given lines.*



Given the lines a and b.

Required *to construct a line x, so that $a : x = x : b$.*

Construction. Draw the indefinite straight line KG and lay off KL equal to a and LR equal to b. Bisect KR at O. With O as a center and OK as a radius, construct semicircle KR. Construct a perpendicular to KR at L, cutting the semicircle at H. HL is the required line.

HINTS. Draw KH and HR and use § 282.

EXERCISES

200. Construct x if $x = \sqrt{ab}$, a and b being lines of given length.

201. Construct a line $\sqrt{6}$ inches long.

HINTS. Let x be the line. Then $x^2 = 6$. Therefore $2 : x = x : 3$.

202. Construct a line $a\sqrt{3}$ inches long, a being a line of given length.

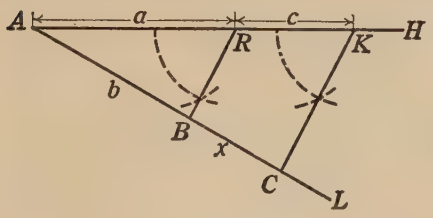
HINTS. Let x be $a\sqrt{3}$. Then $x^2 = 3a^2 = a(3a)$. $a : x = x : 3a$, etc.

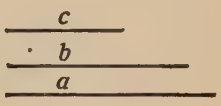
203. Construct x if x is $a\sqrt{\frac{3}{5}}$, a being a given line.

204. Construct x if x is $\frac{a}{2}\sqrt{3}$, a being a given line.

Construction 2

305. *Construct a fourth proportional to three given lines.*





Given the lines a, b, **and** c.

Required *to construct a line* x, *so that* $a : b = c : x$.

Construction. Draw AH and AL, making angle A about 30°. On AH lay off AR equal to a, on RH lay off RK equal to c, and on AL lay off AB equal to b. Draw BR, and through K construct KC parallel to BR, cutting AL at C. Then BC is the required fourth proportional.

The proof should be supplied by the student.

Exercises

205. Construct x if x is equal to $\frac{ab}{c}$, a, b, and c being lines of given length.

206. Construct x if x is equal to ab, a and b being lines of given length.

Hint. $x \cdot 1 = a \cdot b$, etc.

207. Construct a third proportional to a and b, two given lines.

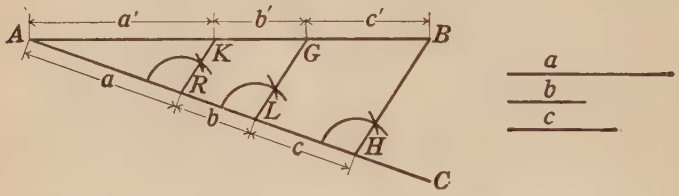
208. Construct x if $a^2 = bx$, a and b being two given lines.

209. Construct x if $a : b = x : c$ where a, b, and c are lines of given length.

210. Construct x if $\frac{a + x}{a - x} = \frac{b}{c}$ where a, b, and c are lines of given length.

Construction 3

306. *Divide a given line into parts proportional to two or more given lines.*



Given the lines *a*, *b*, and *c*, and the line *AB*.

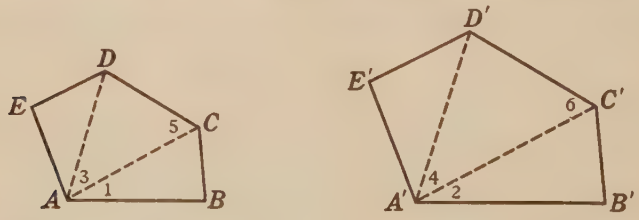
***Required** to divide AB into parts proportional to a, b, and c.*

Construction. Draw AC, making $\angle A$ about 30°. From A on AC lay off AR, RL, and LH equal respectively to a, b, and c. Draw BH, and through R and L construct parallels to BH, cutting AB at K and G respectively. Then AK, KG, and GB are the required parts of the line AB.

The proof is left to the student.

Construction 4

307. *Upon a given side homologous to a side of a given polygon construct a polygon similar to the given one.*



Given the polygon *ABCDE* and the line *A'B'*.

***Required** to construct a polygon similar to ABCDE with AB and A'B' corresponding sides.*

Construction (outline of method only). Draw all the diagonals from A. Then construct $\angle 2 = \angle 1$, $\angle B' = \angle B$, and produce the sides of $\angle 2$ and $\angle B'$ until they meet in C'.

Then construct $\angle 4 = \angle 3$, $\angle 6 = \angle 5$, etc.

The remainder of the construction and the proof are left to the student.

Constructions involving Proportion

211. Given the perimeter, construct a triangle similar to a given triangle.

212. Given the perimeter, construct a rectangle similar to a given rectangle.

213. Divide one side of a triangle into two parts proportional to the other two sides.

214. Construct a triangle, given two sides, a and b, and the ratio of a to the third side c, expressed by two other given lines, h and k.

Hint. Find a fourth proportional to h, k, and a.

215. Construct a triangle, given one side, a, the ratio of a to b, and the ratio of a to c.

216. Draw a line parallel to one side of a rectangle, cutting off a rectangle similar to the given one.

217. Inscribe in a given circle a triangle similar to a given one.

Hint. Circumscribe a circle about the given triangle. Join its center to the vertices and study the central angles so formed.

218. Circumscribe about a given circle a triangle similar to a given one.

219. Given one altitude, construct a triangle similar to a given one.

220. Construct a trapezoid similar to but not equal to a given one.

BOOK IV

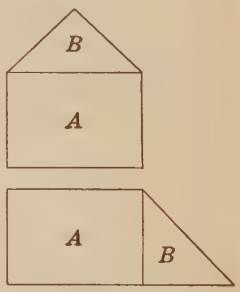
SURFACE MEASUREMENT. AREAS

308. Congruence and equality. It follows from the definition of congruence (§ 24) that congruent figures are equal.

Suppose two plane figures are composed of, or may be divided into, the same number of parts. Then if for each part of the first there is one part of the second congruent to it, and only one, the two figures are equal.

Hence it follows that two equal figures are not always congruent.

Thus the rectangle A and the right triangle B may be put together to form the trapezoid or the pentagon on the right, and these two polygons are equal but not congruent.



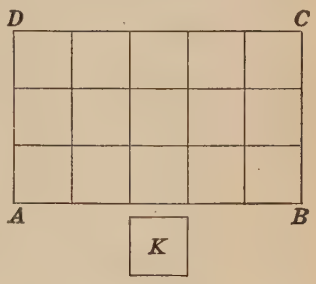
309. Unit of surface. Comparison of the size of plane figures requires measurement of length, since the unit of surface is a square whose side is the unit of length. This square is called the unit square. The unit of surface in practical use is the square inch or the square centimeter, or some multiple of them, as the square yard or the square meter respectively.

310. Area. The *area* of a plane figure is the number which expresses the ratio between its surface and the surface of the unit square.

It is evident that any closed plane surface, whether bounded by straight lines or by curves or in part by both, has an area. To find that area may be easy or it may be difficult. An important

part of the work of plane geometry is finding the area of the simple plane figures.

311. Area of a rectangle. The plane figure the measurement of whose surface most clearly illustrates the notion of area is a rectangle whose sides are exact multiples of the unit of length. Thus, rectangle $ABCD$ is five units long and three units wide. It can, by parallels to the sides, be divided into fifteen squares each equal to the unit square K. We say the area of the rectangle is fifteen square units.



If the length of the base, AB, or that of the altitude, AD, of the rectangle is a mixed number like $5\frac{1}{3}$, the area can still be stated in terms of K and is as before the product of the base by the altitude. This is still true even if the base and the altitude are both mixed numbers.

When the base or the altitude of the rectangle, or both, and the side of the square have no common unit of measure, the area of the rectangle can still be expressed in terms of K and is equal to the product of the base and altitude.

The proof of this last is difficult. It will be omitted, and Theorem 1, of § 312, will be assumed as the basis of all work on area.

Theorem 1

312. *The area of a rectangle is the product of its base and altitude.*

It should be noted that the product of two lines always means the product of their numerical measures.

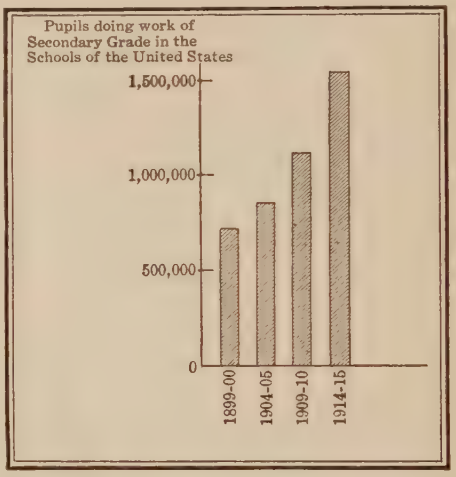
313. ***Corollary 1.*** *The area of a square is the square of one of its sides.*

314. ***Corollary 2.*** *The area of a right triangle is one half the product of the two sides about the right angle.*

315. Example. A report of the Commissioner of Education of the United States shows the number of pupils doing work of secondary-school grade to be as follows:

School year . . .	1899–1900	1904–1905	1909–1910	1914–1915
Pupils (in thousands)	719	876	1131	1587

The above data can be represented graphically by means of rectangles (bars) which have equal bases but whose altitudes vary with the number of pupils enrolled.



Solution. Decide upon a common unit for the base of each rectangle and let one unit of altitude represent 100,000 pupils. We have then the accompanying diagram:

Exercises

1. The following data are taken from the above report:

Year	1890	1895	1900	1905	1910	1915
Public and private high-school pupils studying geometry (in thousands)	59	114	168	219	252	346

Graph the above data as in the foregoing example.

2. The following data are taken from the same report:

Year	1890	1895	1900	1905	1910	1915
Pupils studying algebra (in thousands)	127	245	347	444	465	636

Graph the above data.

3. The same report shows the following percentages of the public and private high-school enrollment to be studying algebra:

Year	1890	1895	1900	1905	1910	1915
Percentages of enrollment studying algebra	43%	52%	55%	56%	57%	49%

Graph the foregoing data.

4. Secure (if possible) and graph data showing for your school the percentage of pupils in each of the grades 9–12, inclusive.

5. The number of guns organized into batteries by the four great allies on November 11, 1918, was as follows:

Nation	France	Italy	England	America
Number of guns (in hundreds)	116	77	70	30

Represent these data graphically.

6. The number of battle planes in each army at the date of the armistice was as follows:

Nation	France	Germany	England	Italy	America	Austria	Belgium
Number of planes (in hundreds)	33	27	18	8	7	6	1.5

Represent these data graphically.

316. Equivalence. If two plane figures have the same area, they are said to be *equivalent*. Equivalence is denoted by the symbol $\fallingdotseq$. Congruent figures, since they can be made to coincide, are equivalent.

The word *equivalent* when applied to areas conveys the same meaning as the word *equal*.

The words *triangle*, *rectangle*, etc. are often used for the area of a triangle, the area of a rectangle, etc.

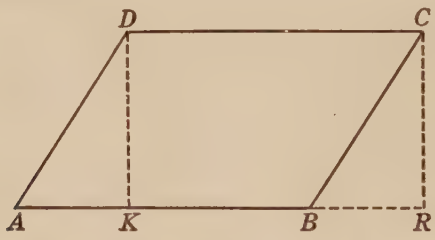
317. Base. Either one of two adjacent sides of a parallelogram can be considered its *base.*

318. Altitude. An *altitude* of a parallelogram is a line drawn from any point in one base perpendicular to the opposite side.

A parallelogram has two bases and two altitudes. By "the base and the altitude of a parallelogram" we mean either base and the *corresponding* altitude. Similarly, "the base and the altitude of a triangle" means any side and the corresponding altitude.

Theorem 2

319. *The area of a parallelogram is the product of its base and altitude.*



Given the parallelogram *ABCD*, and *DK* its altitude on the base *AB*.

To prove *that the area of* $ABCD = AB \times DK$.

Proof. Extend AB and draw the altitudes CR and DK.

$\triangle ADK = \triangle BCR$. Why?

Taking $\triangle BCR$ from quadrilateral $ARCD$ leaves the parallelogram $ABCD$. Why?

Taking $\triangle ADK$ from quadrilateral $ARCD$ leaves the rectangle $KRCD$. Why?

Then parallelogram $ABCD =$ rectangle $KRCD$. § 51

The area of $KRCD = DK \times DC$. § 312

But $DC = AB$. Why?

Therefore the area of the parallelogram $ABCD = AB \times DK$.

Exercises

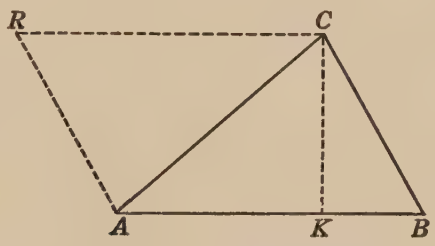
7. The area of a rectangle is 192 square feet and its base is three times its altitude. Find the base and the altitude.

8. The sides of a parallelogram are 10′ and 20′ respectively and one angle is 45°. Show that the area is 141.42 + square feet.

Note. The notation ′ and ″ has long stood for minutes and seconds respectively of an arc. In building-plans and engineers' drawings, however, it is widely used to designate feet and inches respectively. In many of the numerical exercises this notation will be used.

Theorem 3

320. *The area of a triangle is one half the product of its base and altitude.*



Given the triangle *ABC* and its altitude *CK*.

To prove *that the area of* $\triangle ABC = \frac{AB \times CK}{2}$.

Proof. Through C draw a line parallel to AB, and through A draw a line parallel to BC intersecting the first in R.

$\triangle ABC = \triangle ARC$. Why?

Therefore the area of $\triangle ABC = \frac{1}{2}$ the area of parallelogram $ABCR$.

But the area of $ABCR = AB \times CK$. § 319

Therefore the area of $\triangle ABC = \frac{AB \times CK}{2}$.

Exercises

9. Two sides of a triangle are 12″ and 15″ respectively, and their included angle is 30°. Find the area.

10. Find the altitude and the area of an equilateral triangle whose side is 11.

11. Find the side and the area of an equilateral triangle whose altitude is 15.

12. The sides of a triangle are 40′, 50′, and 60′ respectively. Draw the triangle to scale and determine graphically its three altitudes. Using each altitude, compute the area and compare the three results.

Hint. Let 1 inch represent 10 feet.

13. The triangle ABC and the triangle ABK have a common base, AB, and are on the same side of it. The line CK is parallel to AB. Prove the two triangles equivalent.

14. KR is parallel to AB of triangle ABC and cuts AC in K and BC in R. AR and BK are drawn. Prove that the triangle ARC is equivalent to the triangle BKC.

15. $ABCD$ is a quadrilateral and K the mid-point of AD. KR, parallel to AB, and KL, parallel to CD, cut BC in R and L respectively. AR, BK, CK, and DL are drawn. Name the pairs of equivalent triangles thus formed. Prove the triangles of each pair equivalent.

16. The hypotenuse of a right triangle is 58. Another side is 40. Find the third side and the area of the triangle.

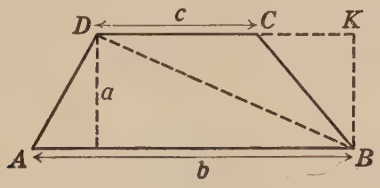
321. Altitude of a trapezoid. The *altitude of a trapezoid* is the perpendicular drawn from any point in one base to the other, produced if necessary.

In the figure on the opposite page, a is the altitude of the trapezoid $ABCD$. Again, BK is also the altitude of $ABCD$.

Query. Are all altitudes of a given trapezoid equal?

Theorem 4

322. *The area of a trapezoid is one half the product of its altitude and the sum of its bases.*



Given the trapezoid *ABCD* in which *b* and *c* are the bases and *a* is the altitude.

To prove *that the area of trapezoid* $ABCD = \frac{a(b+c)}{2}$.

Proof. Draw the diagonal BD and the altitude BK.

$BK = a$. Why?

Now $\triangle ABD = \frac{a \times b}{2}$, (1) Why?

and $\triangle DCB = \frac{BK \times c}{2} = \frac{a \times c}{2}$. (2) Why?

But the trapezoid $ABCD = \triangle ABD + \triangle DCB$. (3)

Therefore, from (1), (2), and (3), the trapezoid

$$ABCD = \frac{a \times b}{2} + \frac{a \times c}{2} = \frac{a(b+c)}{2}.$$

Exercises

17. Find the area of a trapezoid whose bases are 46 inches and 42 inches respectively, and whose altitude is 2 feet.

18. Find the area of the trapezoid $ABCD$ in which the base AB is 100″, AD is 20″, $\angle A$ is 30°, and $\angle B$ is 45°.

19. The bases of a trapezoid are 20′ and 29′ respectively, and the nonparallel sides are 17′ and 10′ respectively. Find the area of the trapezoid.

20. Find the area of a triangle in which two sides are 10″ and 12″ respectively, and their included angle is 45°.

21. In triangle ABC, c is 16′, b is 20′, and $\angle A$ is 120°. Find the area of the triangle.

22. Find the area of a parallelogram in which two adjacent sides are 3′ 4″ and 2′ 3″ respectively, and their included angle is 150°.

23. Find the area of a right triangle in which the hypotenuse and one side are 53″ and 28″ respectively.

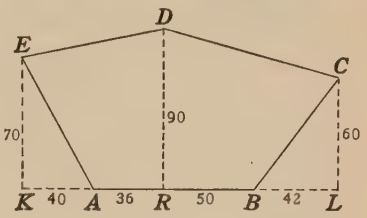
24. The area of an equilateral triangle is $100\sqrt{3}$. Find its side.

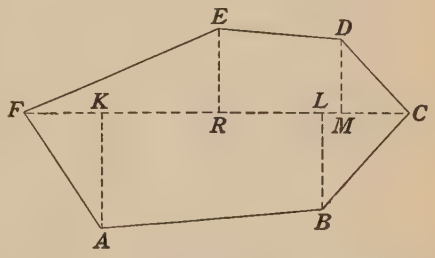
323. Area of an irregular polygon. It is practically impossible to express the area of an irregular polygon of four or more sides in terms of any simply defined bases and altitudes. In such cases the area really depends on the lengths of the sides *and the angles* which the adjacent sides form. The solution of all such problems is fully treated in trigonometry. Certain special cases, however, can be solved by the principles of geometry; and the most general case itself can be solved graphically. If the polygon is a large one, such as a field, it becomes an important matter to make the measurements of sides and angles as few as possible. A practical method in such cases consists in dividing the polygon into triangles and trapezoids and measuring the bases and the altitudes. This method is illustrated in Exs. 25 and 26, where lines only are measured. Another method of determining the area by line measurement only is by measuring the sides and all the diagonals from one vertex. This really divides the polygon into triangles, in each of which the three sides are known. Their areas can then be computed graphically or by Problem 1, § 334. Ex. 27 illustrates the computation of the area of an irregular polygon from measurements of its sides and its angles only.

Exercises

25. In the first figure the dimensions are in rods, the angles at K, R, and L being right angles. Find the area of $ABCDE$ in acres.

26. In the second figure, AK is 60 rods, BL is 50 rods, DM is 40 rods, ER is 45 rods, FK is 42 rods, KR is 60 rods, RL is 54 rods, LM is 10 rods, and MC is 35 rods. The angles at K, R, L, and M are right angles. Find the area of $ABCDEF$ in acres.



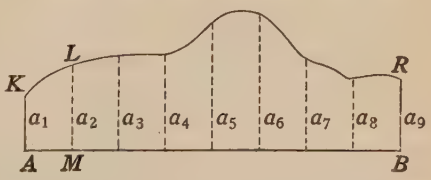


27. In a field $ABCDE$, AB is 50 rods, BC is 35 rods, CD is 85 rods. The angles A, B, C, and D are respectively 110°, 120°, 100°, and 115°. Draw the polygon to scale and compute AE, DE, and the area graphically.

324. Trapezoidal rule. The area bounded by a curved line, a straight line, and two perpendiculars to the latter can be found approximately as follows:

Divide AB into a number of equal parts and measure the length of the perpendiculars from the points to the curve. Then consider each part, as $AMLK$, a trapezoid.

The distances a_1, a_2, etc. are called *offsets*.

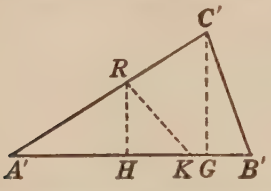


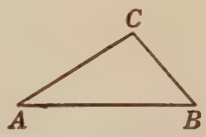
Exercise **28.** Show that the area of a figure like $ABRLK$ can be approximately obtained by the *trapezoidal rule* as follows:

Add half the sum of the first and last offsets to the sum of the other offsets and multiply the result by the distance between the offsets.

Theorem 5

325. *If two triangles have an angle of one equal to an angle of the other, their areas are to each other as the product of the sides including the angle of the first is to the product of the sides including the angle of the second.*





Given the triangles *ABC* and *A'B'C'* in which the angle *A* equals the angle *A'*.

To prove that $\dfrac{\text{area } \triangle ABC}{\text{area } \triangle A'B'C'} = \dfrac{AB \times AC}{A'B' \times A'C'}$.

Proof. Lay off $A'R = AC$ and $A'K = AB$ and draw RK. Draw also altitudes RH and $C'G$.

Then $\triangle ABC = \triangle A'KR$. (1) Why?

Now $\dfrac{\triangle A'KR}{\triangle A'B'C'} = \dfrac{\frac{1}{2}(A'K \times RH)}{\frac{1}{2}(A'B' \times C'G)}$, or $\dfrac{A'K}{A'B'} \times \dfrac{RH}{C'G}$. (2) § 320

$\triangle A'HR \sim \triangle A'GC'$. (3) Why?

$\dfrac{RH}{C'G} = \dfrac{A'R}{A'C'}$. (4) Why?

Substituting $\dfrac{A'R}{A'C'}$ for $\dfrac{RH}{C'G}$ in (2) gives

$$\frac{\triangle A'KR}{\triangle A'B'C'} = \frac{A'K \times A'R}{A'B' \times A'C'}. \quad (5)$$

From (1) and (5), $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB \times AC}{A'B' \times A'C'}$.

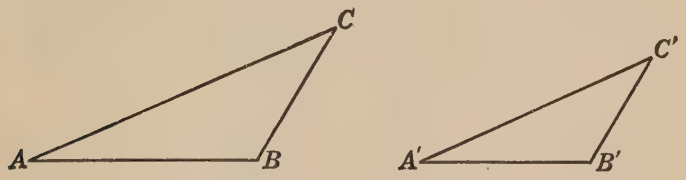
Exercise **29.** Two triangles are equivalent if two sides of one are equal respectively to two sides of the other and the included angles are supplementary.

EXERCISE 30. If two triangles have an angle of one supplementary to an angle of the other, they are to each other as the product of the sides including the angle of the first is to the product of the sides including the angle of the second.

HINT. Place the triangles with the two supplementary angles adjacent.

Theorem 6

326. *The areas of two similar triangles are to each other as the squares of any two homologous sides.*



Given the similar triangles *ABC* and *A'B'C'*.

To prove *that* $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{AC}^2}{\overline{A'C'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}$.

Proof. Since $\angle A = \angle A'$,

$$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB \times AC}{A'B' \times A'C'} = \frac{AB}{A'B'} \times \frac{AC}{A'C'}.$$ (1) § 325

But $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}.$$ (2) Why?

Substituting in (1) for $\frac{AC}{A'C'}$ its equal $\frac{AB}{A'B'}$ obtained from (2) gives

$$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$ (3)

From (2), $$\frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{\overline{AC}^2}{\overline{A'C'}^2}.$$ (4) Why?

From (3) and (4),

$$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{AC}^2}{\overline{A'C'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}.$$

Exercises

31. In triangle ABC side AB is 9″, AC is 10″, and BC is 17″. The area of ABC is 36 square inches. A line parallel to AB cuts BC in K and AC in R so that the area of the triangle CKR is 9 square inches. Find the sides of the triangle CKR.

32. The sides of a triangle are 4″, 5″, and 6″ respectively. Find the sides of a similar triangle whose area is nine times the area of the first.

33. The area of a triangle is 300 square feet and one side is 18′. The corresponding side of a similar triangle is 24′. Find its area.

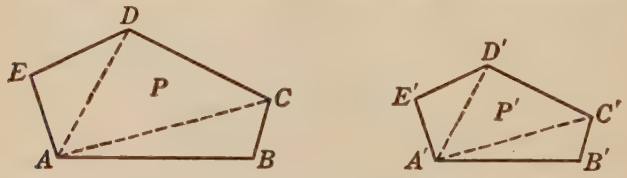
34. A line parallel to AB of the triangle ABC divides it into two parts of equal area. If CB is 10″, find the two parts into which the parallel divides it.

35. K on CB and R on AC of the triangle ABC are points such that the trapezoid $BCRK$ is seven sixteenths of the triangle ABC. If CB is 20 centimeters, find AK.

36. One side of a triangle is 20″. Two parallels to another side divide the triangle into three equivalent parts. Find the three segments into which parallels divide the given side.

Theorem 7

327. *The areas of two similar convex polygons are to each other as the squares of any two homologous sides.*



Given the similar convex polygons P and P', in which the sides AB, BC, etc. correspond respectively to the sides $A'B'$, $B'C'$, etc.

To prove that $$\frac{P}{P'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}, \text{ etc.}$$

Proof. From A and A' draw all the diagonals possible.

Then $\triangle ABC \sim \triangle A'B'C'$, $\triangle ACD \sim \triangle A'C'D'$, etc. Why?

By hypothesis, $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$, etc. (1)

Therefore $\frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{\overline{CD}^2}{\overline{C'D'}^2}$, etc. (2) Why?

Now $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$, (3) § 326

$\frac{\triangle ACD}{\triangle A'C'D'} = \frac{\overline{CD}^2}{\overline{C'D'}^2}$, (4) Why?

and $\frac{\triangle ADE}{\triangle A'D'E'} = \frac{\overline{DE}^2}{\overline{D'E'}^2}$. (5) Why?

From (2), (3), (4), (5), etc.,

$$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\triangle ACD}{\triangle A'C'D'} = \frac{\triangle ADE}{\triangle A'D'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}, \text{ etc.} \quad (6)$$

From (6),

$$\frac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle A'B'C' + \triangle A'C'D' + \triangle A'D'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}, \text{ etc.} \quad \text{§ 276}$$

Therefore $\frac{P}{P'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}$, etc.

Exercises

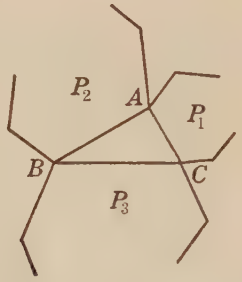
37. The corresponding sides of two similar polygons are 10′ and 24′ respectively. The area of the first is 1352 square feet. Find the area of the second.

38. The areas of two similar polygons are 100 and 256 respectively. One side of the first is 8. Find the corresponding side of the second.

39. The corresponding sides of two similar polygons are 14′ and 48′ respectively. Find their area if the area of a similar polygon equivalent to their sum is 2500 square feet.

Theorem 8

328. *If on the three sides of a right triangle as the homologous sides similar polygons of n sides be constructed, the area of the polygon on the hypotenuse equals the sum of the areas of the polygons on the other two sides.*



Given the right triangle ABC and the similar polygons P_1, P_2, and P_3 described on AC, AB, and BC respectively as the homologous sides.

To prove that $P_3 = P_1 + P_2$.

Proof. $\dfrac{P_1}{P_3} = \dfrac{\overline{AC}^2}{\overline{BC}^2}$, (1) § 327

and $\dfrac{P_2}{P_3} = \dfrac{\overline{AB}^2}{\overline{BC}^2}$. (2) Why?

(1) + (2), $\dfrac{P_1 + P_2}{P_3} = \dfrac{\overline{AC}^2 + \overline{AB}^2}{\overline{BC}^2}$. Why?

But $\dfrac{\overline{AC}^2 + \overline{AB}^2}{\overline{BC}^2} = 1$. Why?

Therefore $P_3 = P_1 + P_2$.

QUERY. Is the theorem of § 284 a special case under Theorem 8 above?

Exercises

40. The corresponding sides of two similar polygons are 10 meters and 24 meters respectively. What is the corresponding side of a polygon similar to the first two and equivalent to their sum?

41. The corresponding sides of two similar polygons are 70 centimeters and 2 meters respectively. Find the corresponding side of a similar polygon equivalent to their difference.

42. The corresponding sides of two similar polygons are 1′ and 1′ 4″ respectively. Find the corresponding side of a polygon similar to them whose area is (1) four times their combined areas; (2) seven times their combined areas.

329. Projection. The projection of one line-segment upon another is the segment of the second line included between the perpendiculars drawn from the extremities of the first line to the second, produced if necessary.

In Figs. 1, 2, and 3 are shown the projections of a line KR on a second line AB. In Fig. 1 the projection is LM, in Fig. 2

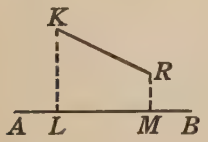


Fig. 1

Fig. 2

Fig. 3

it is LR, and in Fig. 3 it is LM. These correspond to the three possibilities, (1) the first line-segment may not meet the second, (2) it may meet the second, (3) it may cross the second.

Query 1. Can the projection of a line-segment be greater than the line-segment itself? Explain. Can it be less? Can it be equal to it? Explain.

Query 2. Can the projection of a line-segment be zero? Explain.

QUERY 3. In the triangle ABC the line CK is drawn perpendicular to AB. What is the projection of AC on AB? of BC on AB?

QUERY 4. In an obtuse-angled triangle ABC, altitudes AK, BR, and CL are drawn. What is the projection of AB on AC? AB on BC? BC on AC? BC on AB? AC on BC? AC on AB?

EXERCISES

43. A line 12 inches long makes an angle of 30° with another line. Find the length of the projection of the first line on the second.

44. A line KR is 1′ 8″ long and makes an angle of 45° with a second line AB. Find the length of the projection of KR on AB.

45. A line a inches long makes an angle of $x°$ with KR. Find the length of its projection on KR if x is 30°, 45°, 60°, 90°, 120°, 135°, 150°.

Theorem 9

330. *In any triangle the square of the side opposite an acute angle equals the sum of the squares of the other two sides minus twice the product of one of these sides by the projection of the other side upon it.*

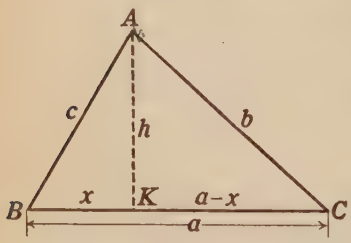


FIG. 1

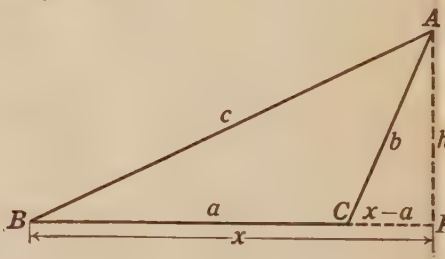


FIG. 2

Given the triangle *ABC*, in which *AK* is the perpendicular to *BC*, segment *BK* is the projection of side *AB* on side *BC*, and the angle *B* is acute.

To prove that $b^2 = c^2 + a^2 - 2ax.$

Proof. In Fig. 1, $b^2 = h^2 + (a - x)^2$. (1) Why?

In Fig. 2, $b^2 = h^2 + (x - a)^2$. (2) Why?

Both (1) and (2) give $b^2 = h^2 + x^2 - 2ax + a^2$. (3)

In Figs. 1 and 2,

$$h^2 + x^2 = c^2. \quad (4)$$

From (3) and (4), $b^2 = c^2 + a^2 - 2ax$. (5)

NOTE. Theorem 9 is an important theorem because of its value in trigonometry. In that subject it is used to compute the angles of any triangle when the lengths of the sides are known. In the right triangle ABK of Fig. 1, $\cos B = \frac{x}{c}$ or $x = c \cos B$. Substituting this value in equation (5) of the proof gives

$$b^2 = c^2 + a^2 - 2ac \cdot \cos B.$$

Similarly, $c^2 = a^2 + b^2 - 2ab \cdot \cos C$.

QUERY. Does Theorem 9 enable one to compute the altitude of a triangle given the three sides? Explain.

EXERCISES

46. In the figures of Theorem 9 draw CR perpendicular at R to BA, produced if necessary, and prove that $b^2 = c^2 + a^2 - 2c \times BR$.

47. If in the figure of Theorem 9 the side AB is $7''$, AC is $15''$, and BC is $20''$, find BK. Then find AK.

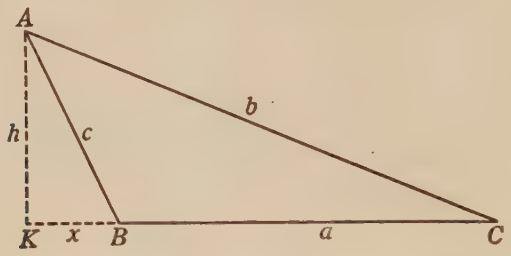
48. The sides of a triangle are $10'$, $17'$, and $21'$ respectively. Find by the use of Theorem 9 the altitude on the side $21'$ and the area of the triangle.

49. The sides of a triangle are $a = 6$, $b = 5$, and $c = 4$. Using the equation given in the note above and the table on page 188, find the approximate values of the angles of the triangle.

50. From an inspection of the two formulas of the note above state a similar formula which would involve $\cos A$.

Theorem 10

331. *In an obtuse-angled triangle the square of the side opposite the obtuse angle equals the sum of the squares of the other two sides plus twice the product of one of these sides by the projection of the other side upon it.*



Given the triangle *ABC*, in which *AK* is the perpendicular to *BC*, *BK* is the projection of *AB* on *BC*, and the angle *B* is obtuse.

To prove that $b^2 = c^2 + a^2 + 2\,ax.$

Proof. $b^2 = h^2 + (x + a)^2,$ (1) Why?

or $b^2 = h^2 + x^2 + 2\,ax + a^2.$ (2) Why?

Now $h^2 + x^2 = c^2.$ (3) Why?

From (2) and (3), $b^2 = c^2 + a^2 + 2\,ax.$

Exercises

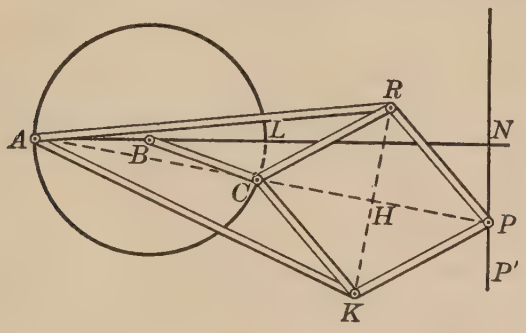
51. In the figure of Theorem 10 draw CR perpendicular to AB produced at R and prove that $b^2 = a^2 + c^2 + 2\,AB \times BR.$

52. The sides of a triangle are 18″, 20″, and 34″ respectively. Find by the use of Theorem 10 the altitude on the side 18″ and the area of the triangle.

53. The sides of a triangle are 12′, 17′, and 25′ respectively. Find the altitude on the side 25′ and the area.

54. If the angle C of the triangle ABC is 120°, prove that $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + AC \times BC.$

332. Peaucellier's linkage. The figure below gives a top view of a linkage, sometimes called compound compasses, designed to draw a straight line without a ruler. This form of the instrument was invented by A. Peaucellier of Nice. He published a description of his instrument in 1873. Some time afterward he was awarded for his invention the "Prix Montyou," the great mechanical prize of the Institute of France. In the figure, $CKPR$



is a rhombus, $AR = AK$, $AB = BC$, and points A and B are fixed. The bars meeting at the six points A, B, C, R, P, and K are hinged. Hence, if the rhombus be moved, R and K will describe a circle about A, C will move in a circle about B, and P, carrying a pencil, will move in a straight line. The dotted lines are used only in the proof which follows.

Points A, C, and P lie in a straight line, the perpendicular bisector of KR.

$$\overline{AR}^2 = \overline{AC}^2 + \overline{CR}^2 + 2\,CH \cdot AC. \quad \text{Why?} \qquad (1)$$

$$\overline{AR}^2 - \overline{CR}^2 = AC(AC + 2\,CH) \qquad (2)$$

$$= AC \cdot AP. \qquad (3)$$

If PN is $\perp$ to AL, $$\frac{AL}{AP} = \frac{AC}{AN}. \qquad (4)$$

Therefore $$AL \cdot AN = AC \cdot AP. \qquad (5)$$

From (5) and (3), $$AL \cdot AN = (\overline{AR}^2 - \overline{CR}^2). \qquad (6)$$

Whence $$AN = (\overline{AR}^2 - \overline{CR}^2) \div AL. \qquad (7)$$

Since AR, CR, and AL are fixed lengths, the length of AN is constant. Therefore, from any position of P a $\perp$ drawn to AL will always meet it in the same point N; that is, P will move in a straight line.

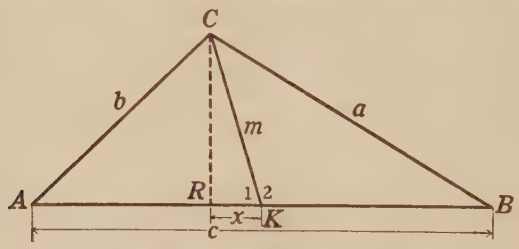
NOTE. The proofs of Theorems 9 and 10 illustrate the fact that a demonstration is more easily followed if the lines involved are denoted by one letter instead of two. It is often a great help in solving exercises to simplify the notation in this way.

QUERY. How can one determine from the three sides of a triangle whether it has a right angle? an obtuse angle? an acute angle?

EXERCISE 55. The sides of a triangle in which C is an acute angle are a, b, and c respectively. Show that the projection of the side a on the side b is $\dfrac{a^2+b^2-c^2}{2b}$, and that the altitude on the side b is $\sqrt{a^2-\left(\dfrac{a^2+b^2-c^2}{2b}\right)^2}$.

Theorem 11

333. *In any triangle the sum of the squares of two sides equals half the square of the third side plus twice the square of the median drawn to the third side.*



Given the triangle *ABC*, in which *CK* is the median to *AB*.

To prove that $a^2 + b^2 = \dfrac{c^2}{2} + 2m^2$.

Proof. Draw $CR \perp$ to AB.

If CR coincides with CK, $\triangle ABC$ is isosceles and the truth of the theorem immediately follows. If CR and CK do not coincide, then $\angle 1$ and $\angle 2$ will be unequal. Let $\angle 2$ be obtuse.

Now $AK = KB = \frac{c}{2}$. (1) Why?

Then, in $\triangle BCK$, $a^2 = \left(\frac{c}{2}\right)^2 + m^2 + 2\left(\frac{c}{2}\right)x$, (2) § 331

and, in $\triangle ACK$, $b^2 = \left(\frac{c}{2}\right)^2 + m^2 - 2\left(\frac{c}{2}\right)x$. (3) § 330

(2)+(3), $a^2 + b^2 = \frac{c^2}{2} + 2m^2$.

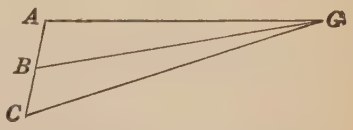
Exercises

56. The sides of a triangle are 16 centimeters, 30 centimeters, and 34 centimeters respectively. Find the median drawn to the longest side.

57. The sides of a triangle are 1 meter 40 centimeters, 1 meter 60 centimeters, and 2 meters respectively. Find to two decimals the length of the median to the side 1 meter 40 centimeters.

58. ABC is a right triangle and CK the median on the hypotenuse. Prove that $4\overline{CK}^2 = \overline{AB}^2$.

Note. Determining the exact position of the big guns of the enemy so that they can be destroyed by counter fire is an important part of modern warfare. One way of doing this is to have at each of three or more suitably placed stations certain instruments by which the report of the big gun is noted and also the instant at which the sound arrives at each station. A comparison of results shows the precise time-difference in the arrival of the sound at the three stations. From the thermometer and barometer readings the velocity of sound can be computed. From this data the position of the gun can then be calculated.



59. Three stations A, B, and C are in a straight line so that $AB = BC = 2$ miles. The report of a big gun reaches B one second after it reaches A and reaches C two seconds after it reaches B. The velocity of sound is 1120 feet per second. How far is the gun from A, B, and C?

HINTS. Let t = the number of seconds sound requires to travel from G to A and let v equal the velocity of sound in miles per second. Then $GA = vt$, $GB = v(t+1)$, and $GC = v(t+3)$.

By § 333, $\overline{GA}^2 + \overline{GC}^2 = 2\,\overline{GB}^2 + 2\,\overline{AB}^2$. (1)

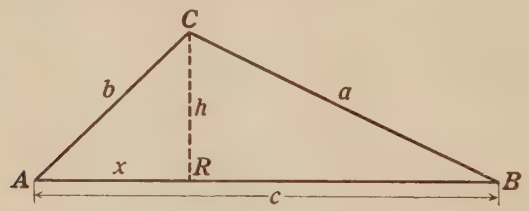
Substituting in (1) and solving,

$$vt = \frac{4}{v} - \frac{7v}{2} = \frac{4}{\frac{7}{33}} - \frac{7\left(\frac{7}{33}\right)}{2} = 18.1131 \text{ miles} = 18 \text{ miles } 605 \text{ feet etc.}$$

EXERCISE 60. In the preceding exercise draw the triangle ACG to scale and determine by means of a protractor the angle GAC.

Problem 1 (*Computation*)

334. *Compute the area of a triangle in terms of its three sides.*



Given the triangle *ABC* in which the angle *A* is acute.

To compute *the area, T, of $\triangle ABC$ in terms of the sides a, b, and c.*

Solution. Draw altitude CR.

Then $T = \frac{ch}{2}$. (1) § 320

But $h = \sqrt{b^2 - x^2}$. (2) Why?

Therefore (1) becomes

$$T = \frac{c\sqrt{b^2 - x^2}}{2}.$$ (3) Why?

Now since $\triangle ABC$ must have two acute angles, if $\angle A$ is acute, then $a^2 = b^2 + c^2 - 2cx$. (4) § 330

Solving (4) for x, $\quad x = \dfrac{b^2 + c^2 - a^2}{2c}$. (5)

Substituting this value for x in (3),

$$T = \frac{c}{2}\sqrt{b^2 - \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2}. \quad (6)$$

Equation (6) expresses the area of a triangle in terms of its sides, but in a form extremely inconvenient to use for numerical computation. It can be put in a form better adapted for such a purpose as follows:

$$T = \frac{c}{2}\sqrt{\left(b + \frac{b^2 + c^2 - a^2}{2c}\right)\left(b - \frac{b^2 + c^2 - a^2}{2c}\right)} \quad \text{Why?} \quad (7)$$

$$= \frac{c}{2}\sqrt{\left(\frac{2bc + b^2 + c^2 - a^2}{2c}\right)\left(\frac{2bc - b^2 - c^2 + a^2}{2c}\right)} \quad (8)$$

$$= \frac{c}{2}\sqrt{\frac{[(b^2 + 2bc + c^2) - a^2][a^2 - (b^2 - 2bc + c^2)]}{4c^2}} \quad (9)$$

$$= \frac{1}{4}\sqrt{[(b + c)^2 - a^2][a^2 - (b - c)^2]} \quad (10)$$

$$= \frac{1}{4}\sqrt{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}. \quad (11)$$

In order to simplify (11) still further, let

$$a + b + c = 2s. \quad (12)$$

$$2a = 2a. \quad (13)$$

(12)−(13), $\quad -a + b + c = 2s - 2a = 2(s - a)$. (14)

In like manner $\quad a + b - c = 2(s - c)$, (15)

and $\quad a - b + c = 2(s - b)$. (16)

Substituting from (12), (14), (15), and (16) in (11), the latter becomes

$$T = \tfrac{1}{4}\sqrt{2s \times 2(s - a)\,2(s - c)\,2(s - b)} \quad (17)$$

$$= \tfrac{1}{4}\sqrt{16\,s(s - a)(s - b)(s - c)} \quad (18)$$

Finally, $\quad \boldsymbol{T = \sqrt{s(s - a)(s - b)(s - c)}}$. (19)

Example. Compute the area of a triangle whose sides are 9″, 10″, and 17″ respectively.

Solution. $T = \sqrt{s(s-a)(s-b)(s-c)}$.

$a + b + c = 2s = 9 + 10 + 17 = 36$. Whence $s = 18$.

Then $T = \sqrt{18(18-9)(18-10)(18-17)}$

$= \sqrt{18 \cdot 9 \cdot 8 \cdot 1} = \sqrt{3^2 \times 2 \times 3^2 \times 2^3}$

$= \sqrt{3^4 \times 2^4} = 3^2 \times 2^2 = 36$.

Note that s is not the sum of the three sides, but *half the sum*.

Query. In the preceding example what is the easiest method of finding one or all of the altitudes?

Miscellaneous Numerical Exercises

In the following exercises, whenever possible, use equation 19 of § 334:

61. Find the area of a triangle whose sides are respectively 7′, 15′, and 20′.

62. Find the area of a triangle whose sides are respectively 15″, 28″, and 41″.

63. Find the altitudes of the triangle whose sides are 35, 52, and 73 respectively.

64. Find the area and the altitude on the side 200 of a triangle whose sides are 87, 119, and 200 respectively.

65. Find the radius of the circumscribed circle of a triangle whose sides are 116, 231, and 325 respectively.

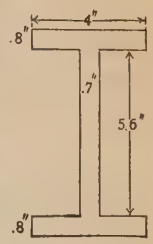
Hint. See § 294.

66. Find the area of a trapezoid whose sides are 10 meters, 17 meters, 30 meters, and 51 meters respectively, the last two being the bases.

67. $ABCDE$ is a pentagon. The lengths in yards of AB, BC, CD, DE, and EA are respectively 60, 111, 39, 75, and 117. The diagonal AC is 153 yards and AD is 120 yards. How many acres are there in the pentagon?

68. How many tiles 4 inches square will it take to cover the floor of a hall 60′ 4″ long and 15′ 8″ wide?

69. A floor is 14′ 6″ by 18′ 3″. How many tiles in the shape of a regular hexagon, each side being one half an inch, will be required to cover it if the cement between the tiles covers one twelfth of the floor space?



70. Find the area of the cross section of an I-beam with dimensions as given in the adjacent figure.

71. The figure below represents a map of Kansas. To what scale is it drawn if a line from the southeast corner to the northwest corner is 440 miles long? Compute the distance from Wichita to Kansas City. Compute approximately the area of the state.

72. Compute from the figure above by means of the scale the perimeter and the area of New Mexico.

73. The area of a right triangle is 30 square feet. The hypotenuse is 13 feet. Find the other two sides.

HINTS. Let x and y represent the sides about the right angles respectively.

Then $$x^2 + y^2 = 169, \qquad (1)$$

and $$\frac{xy}{2} = 30. \qquad (2)$$

From (2), $$2xy = 120. \qquad (3)$$

From (1) and (3), $$x^2 + 2xy + y^2 = 289, \qquad (4)$$

and $$x^2 - 2xy + y^2 = 49. \qquad (5)$$

From (4), $$x + y = \pm 17.$$

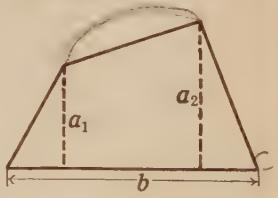
From (5), $$x - y = \pm 7 \text{ etc.}$$

74. The area of a right triangle is $6\frac{2}{3}$ square yards and the hypotenuse is 17′. Find the other two sides.

75. The homologous sides of two similar polygons are 35′ and 12′ respectively. Find the homologous side of a similar polygon whose area is sixteen times the sum of the areas of the first two.

Historical Note. The discovery of the proof of the formula of § 334 is credited to Hero of Alexandria (about 100 B.C.). Hero's proof of the formula for the area of a triangle is given in Ball's "History of Mathematics," pp. 92–93. Though very different from the proof here given, being more geometrical in character, it is easily understood. It is remarkable that a mind so able could not derive the correct formula for the area of a quadrilateral. His formula is

$$\left(\frac{a_1 + a_2}{2}\right)\frac{b}{2},$$



in which a_1, a_2, and b are defined as in the adjacent figure.

Hero's activities extended to astronomy, surveying, and physics as well as to mathematics. Eighteen centuries before Watt, Hero devised an engine with steam as the motive power, which ran on the same principle as that of a rotary lawn-sprinkler.

Miscellaneous Exercises

76. A median of a triangle divides it into two equivalent parts.

77. H is any point on the diagonal AC of the parallelogram $ABCD$. Draw DH and BH and prove that the triangles ABH and ADH are equivalent.

78. R is any point on the median AK of the triangle ABC. Draw BR and CR and prove the triangles ABR and ACR equivalent.

79. The line joining the mid-points of the bases of a trapezoid divides it into two equivalent parts.

80. K is any point on the side AD and R any point on the side DC of the parallelogram $ABCD$. Draw lines AR, BR, KB, and KC and prove that the triangles BKC and ARB are equivalent.

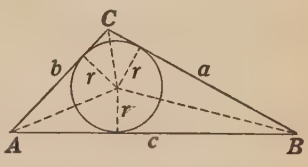
81. K is any point within the parallelogram $ABCD$. Draw lines from K to the four vertices and prove that the sum of the areas of the triangles KAB and KCD is one half the area of the parallelogram.

82. The mid-points of two sides of a triangle are joined to any point in the third side. Prove that the area of the quadrilateral thus formed is one half the area of the triangle.

83. The area of a triangle is one half the product of its perimeter and the radius of its inscribed circle.

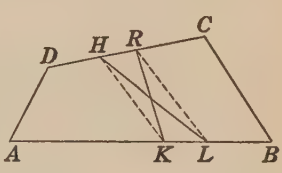
84. The radius of the inscribed circle of a triangle whose sides are a, b, and c respectively is

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$



85. The sum of the perpendiculars from any point within an equilateral triangle to the sides is equal to the altitude of the triangle.

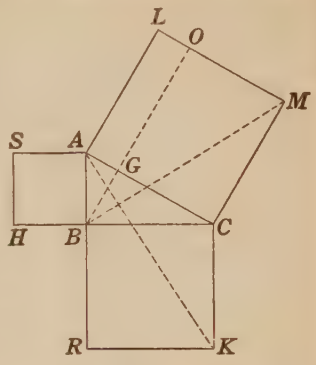
86. $ABCD$ is a quadrilateral divided into two equivalent parts by KR, the point K being on AB and the point R on CD. The point L is on KB. A parallel to LR through K cuts CD (not CD produced) in H. Prove that LH also bisects the quadrilateral.



87. $ABCD$ is a quadrilateral such that K, a point within the triangle ABC, makes the area of the quadrilateral $AKCD$ one half that of $ABCD$. A parallel to AC through K cuts BC in the point R. Draw the line AR and prove that the quadrilateral $ARCD$ is equivalent to the triangle ABR.

88. Parallels are drawn through the vertices of the triangle ABC cutting the sides or the sides produced in K, R, and L respectively. Draw KR, RL, and LK and prove that the area of the triangle KRL is twice that of ABC.

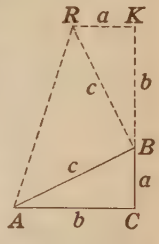
89. R and K are any two points on the sides CD and BC respectively of the parallelogram $ABCD$. A parallel to AK through D cuts AR produced in L. A parallel to AL through K cuts DL produced in M. Prove that $AKML$ and $ABCD$ are equivalent.



90. Prove that the area of the square on the hypotenuse of a right triangle equals the sum of the areas of the squares on the other two sides.

HINTS. In the adjacent figure the triangle ACK is congruent to the triangle BCM. Therefore the rectangle $OMCG$ is equivalent to the square $BCKR$ etc.

91. Prove that, in the right triangle ABC, $a^2 + b^2 = c^2$, using the adjacent figure, which was devised by President Garfield in an original proof of the Pythagorean Theorem.



HISTORICAL NOTE. One of the great teachers of the times preceding Euclid was Pythagoras. After study and travel in various Mediterranean countries he returned to Samos, his birthplace, where his teaching had little success. Later he went to Croton, a prosperous Greek colony in southern Italy, and here great numbers of enthusiastic disciples attended his school. He was a moralist and a philosopher and tried to construct a mathematical basis for his moral and philosophical teachings. Unfortunately the political results of his teaching brought him and his school under suspicion, and in a popular uprising many of his associates were killed and he himself had to flee. A year later, in 500 B.C., he met his death in a similar disturbance in the city of Metapontum.

In all his work the ideal of Pythagoras was high — knowledge, not wealth, being the object of his study. In geometry the discovery and proofs of numerous theorems are due to him. The one of Ex. 90, though known to the Hindus and the Egyptians for centuries before his time, has long been called the Pythagorean Theorem. It seems certain that he was the first to prove it. But which one of the fifty or more proofs now known is his has not been determined.

Problem 2 (*Construction*)

335. *Construct a square equivalent to the sum of two given squares.*

HINT. Study Theorem 8 (§ 328) and devise a method of proof.

Problem 3 (*Construction*)

336. *Construct a polygon equivalent to the sum of two given similar polygons and similar to them.*

HINT. Study Theorem 8 (§ 328) and devise a method of proof.

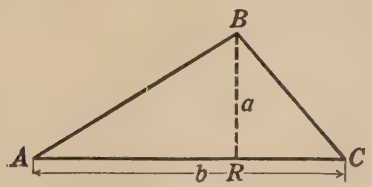
EXERCISES

92. Construct a square equivalent to the difference of two given squares.

93. Construct a square equivalent to the sum of three given squares.

Problem 4 (*Construction*)

337. *Construct a square equivalent to a given triangle.*



Given the triangle *ABC* with base *b* and altitude *a*.

Required *to construct a square equivalent to* $\triangle ABC$.

Analysis. Let x be the side of the required square. Then its area is x^2.

Now the area of $\triangle ABC = \frac{ab}{2}$.

Hence $x^2 = \frac{ab}{2}$.

Then $\frac{a}{2} : x = x : b$. Why?

The construction and proof are left to the student.

NOTE. It should be noted that heretofore any description of the method of thinking out the solution of a construction problem has been carefully excluded from the solution itself. In many problems involving equivalent areas, however, it seems best to include as one step of the detailed explanation the analysis which led to the solution. It is desirable especially with those problems in which a simple equation so clearly and simply indicates what the subsequent steps should be.

EXERCISES

94. Construct a square equivalent to a given parallelogram.

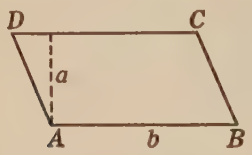
95. Construct a square equivalent to a given trapezoid.

96. Construct a square which is three fifths of a given rectangle.

97. Construct a square which will be to a given square in the ratio of 7 to 4.

Problem 5 (*Construction*)

338. *On a given line as the base, construct a triangle equivalent to a given parallelogram.*



Given *c*, the base of the required triangle, and the parallelogram *ABCD*, whose altitude is *a* and whose base is *b*.

Required *to construct a triangle equivalent to ABCD on c as the base.*

Analysis. Let x be the unknown altitude of the required triangle.

Then its area is $\frac{cx}{2}$. That of the parallelogram is ab.

Then $$\frac{cx}{2} = ab.$$

Hence $$\frac{c}{2} : a = b : x.$$ Why?

The construction and proof are left to the student.

339. Use of an equation in construction problems. Most of the problems of Book IV require the construction of a figure equivalent to a given one or to a certain definite part of a given one. Such problems can be solved most readily as follows:

First, express as an equation the stated or implied relation. Then represent the unknown line by x and express the area of the required figure in terms of x and the given parts.

Then express the area of the given figure in terms of its dimensions.

Lastly, use the conditions stated in the problem and state an equation, as was done in §§ 337 and 338. An inspection of this equation, when put into the form of a proportion, will show whether a mean proportional or a fourth proportional is required to solve the given problem.

It is worth noting that the solution of a construction problem involving areas can very often be made to depend on either a mean proportional or a fourth proportional, or on both.

Exercises

98. On a given base construct a triangle equivalent to a given square.

99. On a given base construct a triangle equivalent to a given triangle.

100. On a given base construct a triangle equivalent to the sum of two given triangles.

Query 1. How many triangles may be constructed which satisfy the conditions of § 338?

Query 2. Is more than one solution possible for the problem of § 335? of § 336? of § 337?

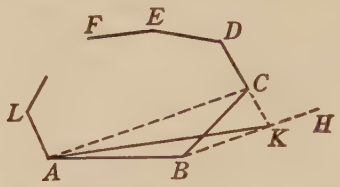
Query 3. How many parallelograms may be constructed equivalent to a given square?

Query 4. How many solutions are possible for the following: On a given base construct a parallelogram equivalent to a given square.

Query 5. How many solutions are possible for the following: Given the two bases and another side, construct a trapezoid equivalent to a given square.

Problem 6 (*Construction*)

340. *Transform a polygon of n sides into an equivalent polygon of $n-1$ sides.*



Given the polygon $ABCDEF\cdots$, having n sides.

Required *to transform the polygon $ABCDEF\cdots$ into an equivalent polygon having $n-1$ sides.*

Construction (outline of method). Draw any diagonal which joins two alternate vertices, as AC.

Through B draw $BH \parallel$ to AC.

Extend DC (or LA) to meet BH at K.

Draw AK.

Then $AKDEF\cdots$ is equivalent to $ABCDEF\cdots$ and has $n-1$ sides.

HINT. $\triangle ACB = \triangle ACK$. Why?

EXERCISES

101. Using the method of Problem 6, transform a quadrilateral into an equivalent triangle.

102. Transform a pentagon into an equivalent triangle.

103. Transform a hexagon into an equivalent triangle.

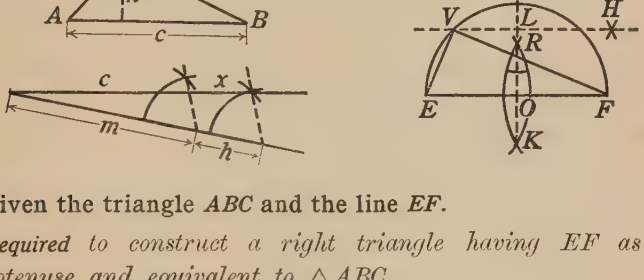
QUERY. How may a polygon of n sides be transformed into an equivalent triangle?

104. Construct a triangle equivalent to the sum of two given triangles.

HINT. Use Problem 6.

NOTE. Brevity, completeness, and precision, or, in one word, conciseness, are desirable qualities in all effective writing; and they are especially desirable in writing out the solution of a construction problem. In the following example observe particularly that all necessary lines are actually constructed and that the arcs for the construction of a perpendicular and those for constructing the needed parallels are drawn. Note how the accompanying diagrams cut down lengthy explanations.

Example. *Given the hypotenuse, construct a right triangle equivalent to a given triangle.*



Given the triangle *ABC* and the line *EF*.

Required *to construct a right triangle having EF as the hypotenuse and equivalent to* $\triangle ABC$.

Analysis. Let c and h be respectively the base and the altitude of the given triangle ABC and let x be the unknown altitude upon the hypotenuse m, which equals EF, of the required triangle.

Then $\frac{ch}{2} = \frac{mx}{2}$. (1) § 320

Hence $m : c = h : x$. (2) § 289

Construction. Construct x, a fourth proportional to m, c, and h. Then construct $KR \perp$ to EF at its mid-point O and describe semicircle EF. On OR lay off OL equal to x. Construct $LH \perp$ to OL, cutting the semicircle in V. Draw EV and FV. Then EVF is the required triangle.

Proof. $\angle EVF = 90°$. Why ?

The altitude of $\triangle EVF$ on $EF = LO = x$.

$\triangle EVF \fallingdotseq \triangle ABC$ by (1) and (2).

QUERY 1. Is a solution of the preceding example possible for all values of m, c, and h? Explain.

QUERY 2. What is the relation between m, c, and h which makes a solution possible?

PROBLEMS OF CONSTRUCTION INVOLVING AREAS

105. Construct a triangle equivalent to a given trapezoid. Discuss (see § 244).

106. Divide a triangle into five equivalent parts by lines drawn from one vertex. Discuss.

107. Construct a square five times as large as a given square. Discuss.

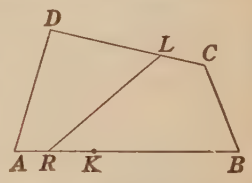
108. Through a given point within a parallelogram draw a line dividing it into two equivalent parts. Discuss.

109. Divide a parallelogram into three equivalent parts by lines through one vertex. Discuss.

110. Given two sides, construct a triangle equivalent to a given triangle. Discuss.

111. Divide a parallelogram into two equivalent parts by a line perpendicular to one side. Discuss.

112. Given one diagonal, construct a rhombus equivalent to a given parallelogram. Discuss.



113. RL in the adjacent figure divides the quadrilateral $ABCD$ into two equivalent parts. Draw through K a line which will also divide $ABCD$ into two equivalent parts. Discuss.

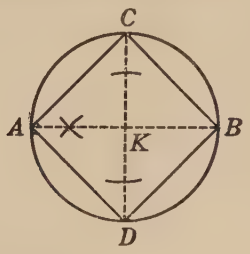
114. Construct an equilateral triangle whose area is twice that of a given equilateral triangle. Discuss.

BOOK V

REGULAR POLYGONS AND THE MEASUREMENT OF THE CIRCLE

Problem 1 *(Construction)*

341. *Inscribe a square in a given circle.*



Given a circle whose center is *K*.

Required *to inscribe a square in the circle.*

Construction. Draw the diameter CD and construct the diameter $AB \perp$ to it. Draw the chords AC, CB, BD, and DA.

Then $DBCA$ is the required square.

The proof is left to the student.

Exercises

In Exs. 1–9 the result obtained from any one may be used, where possible, in any of those following. In problems dealing with regular polygons (see § 79) R, unless otherwise stated, represents the radius of the circumscribed circle.

1. Circumscribe a square about a given circle.
2. Inscribe a regular octagon in a given circle.

3. Inscribe a circle in a given square.

4. Describe a method by which regular polygons of 16 sides, 32 sides, 64 sides, etc. may be inscribed in a circle.

5. If the alternate vertices of a regular polygon of an even number of sides are joined, the polygon thus formed is regular.

NOTE. Problems 1–6 of this book, and the exercises which follow each, deal with certain regular polygons which can be inscribed in a circle and circumscribed about a circle. It will be assumed that for these regular polygons the center of the inscribed circle coincides with the center of the circumscribed circle. Later this assumption will be proved.

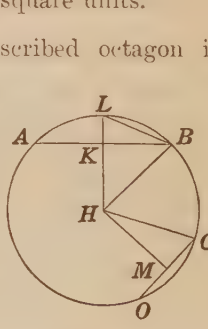
342. Apothem. The *apothem* of a regular polygon is the perpendicular from the center of its circumscribed (or inscribed) circle to any side of the polygon.

EXERCISES

6. Show that the side of an inscribed square is $R\sqrt{2}$, that its apothem is $\frac{R}{2}\sqrt{2}$, and that its area is $2R^2$ square units.

7. Show that the side of a regular inscribed octagon is $R\sqrt{2-\sqrt{2}}$.

HINTS. Let AB in the adjacent figure be the side of an inscribed square and BL the side of the regular inscribed octagon. Compute in terms of R the value of each of the following lines in the order given: AB, BK, KH, KL, and BL.



8. Show that the apothem of a regular inscribed octagon in terms of R is $\frac{R}{2}\sqrt{2+\sqrt{2}}$.

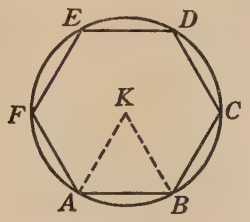
HINTS. In the figure of Ex. 7 let OC equal BL. Then MC is how much? HM is how much?

9. Show that the area of a regular inscribed octagon is $2R^2\sqrt{2}$.

10. A regular octagon is inscribed in a circle whose radius is 10. Show that its side is 7.65 +, its apothem 9.23 +, and its area 282.84 + square units.

Problem 2

343. *Inscribe a regular hexagon in a given circle.*



Given a circle whose center is *K*.

Required *to inscribe a regular hexagon in the circle.*

Construction. Draw any radius KA. Then with A as the center and AK as the radius, describe an arc cutting the circle at B. With B as the center and KA as the radius, describe an arc cutting the circle at C. In like manner obtain points D, E, and F. Draw the chords AB, BC, CD, DE, EF, and FA. They will form the required regular hexagon.

Proof. Draw the radius KB.

Then $\triangle ABK$ is equilateral. Why?

Hence $\angle K = 60°$, Why?

and arc $AB = 60°$, or one sixth of the circumference. Why?

Hence arc $ABCDEF$ is five sixths of the circle, and arc FA is one sixth.

That is, chord FA = chord AB, § 178

and the hexagon $ABCDEF$ is equilateral.

Now $\angle D$ = one half of $240° = 120°$. Why?

In like manner $\angle E$, $\angle F$, etc. are each equal to 120°,

and the hexagon is equiangular.

Therefore the polygon $ABCDEF$ is a regular hexagon inscribed in the circle. § 79

Exercises

In Exs. 11–18 the result of any one may be used, if needed, in any one following.

11. Inscribe an equilateral triangle in a given circle.

12. Inscribe a regular dodecagon in a given circle.

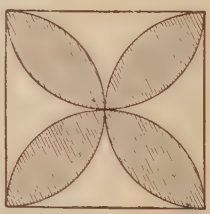
13. Circumscribe a regular hexagon about a given circle.

14. Circumscribe an equilateral triangle about a circle.

15. Show that the apothem of a regular inscribed hexagon is $\frac{R}{2}\sqrt{3}$ and that the area is $\frac{3R^2}{2}\sqrt{3}$.

16. Show that the side of a regular inscribed dodecagon is $R\sqrt{2-\sqrt{3}}$, the apothem $\frac{R}{2}\sqrt{2+\sqrt{3}}$, and the area $3R^2$.

17. Show that the side, the apothem, and the area of a regular inscribed dodecagon in a circle whose radius is 20 inches are respectively 10.35 + inches, 19.31 + inches, and 1200 square inches.



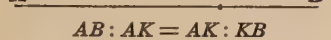
18. Construct the above designs.

19. The radius of a circle is 18. Show that the area of its regular circumscribed hexagon is 1122.36 + square units.

20. Show that the side, the apothem, and the area of an inscribed equilateral triangle are respectively $R\sqrt{3}$, $\frac{R}{2}$, and $\frac{3R^2\sqrt{3}}{4}$.

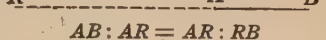
21. Show that the side, the apothem, and the area of a circumscribed equilateral triangle are respectively $2R\sqrt{3}$, R, and $3R^2\sqrt{3}$.

344. Mean and extreme ratio. A line is divided by a point in *mean and extreme ratio* when one segment is a mean proportional between the whole line and the other segment.



$AB : AK = AK : KB$

FIG. 1



$AB : AR = AR : RB$

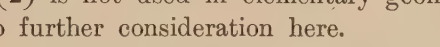
FIG. 2

There are two cases: (1) when the point K is on the given line AB (Fig. 1); (2) when the point R is on the given line BA produced (Fig. 2).

As case (2) is not used in elementary geometry, it will be given no further consideration here.

Problem 3 (*Computation*)

345. *Express the numerical values of the two segments of a line of length a divided internally in mean and extreme ratio by a point on the line.*



Given the line AB of length a, and the point K dividing the line in extreme and mean ratio.

Required *to express the values of the segments AK and KB in terms of a.*

Solution. Let K be the point which divides AB internally in mean and extreme ratio. Then, by the definition of § 344,

$$AB : AK = AK : KB. \tag{1}$$

Let $AK = x.$

Then $KB = a - x.$

Substituting in (1), $a : x = x : (a - x).$ (2)

Then $$x^2 = a^2 - ax. \tag{3}$$

$$x^2 + ax = a^2. \tag{4}$$

Completing the square,

$$x^2 + ax + \frac{a^2}{4} = a^2 + \frac{a^2}{4} = \frac{5\,a^2}{4}. \tag{5}$$

$$x + \frac{a}{2} = \pm \frac{a}{2}\sqrt{5}. \tag{6}$$

Therefore $$x = -\frac{a}{2} + \frac{a}{2}\sqrt{5}, \text{ or } \frac{a}{2}(-1+\sqrt{5}), \tag{7}$$

and $$x = -\frac{a}{2} - \frac{a}{2}\sqrt{5}, \text{ or } \frac{a}{2}(-1-\sqrt{5}). \tag{8}$$

Since (8) gives a *negative* value of x, it is neither AK nor BK of the figure. Therefore (7) gives the length of AK.

Hence $$BK = a - x = a - \left(-\frac{a}{2} + \frac{a}{2}\sqrt{5}\right) = \frac{3\,a}{2} - \frac{a}{2}\sqrt{5},$$

or $$BK = \frac{a}{2}(3 - \sqrt{5}). \tag{9}$$

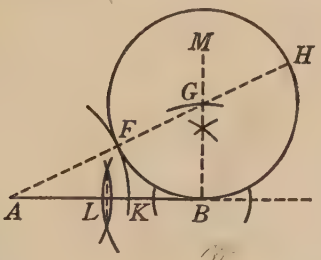
EXERCISE **22.** Show that the segments of a line $12'$ long divided internally in mean and extreme ratio are $7.42'-$ and $4.57'+$ respectively.

NOTE. The problem of § 345 was of great interest to the early Greek geometers. Plato (429–347 B.C.) was the first to discuss and solve the problem. Later Eudoxus (408–355 B.C.), who is usually ranked as third among the Greek mathematicians, added several other theorems concerning it. Euclid in the thirteenth book of his "Elements" has five theorems on the "golden section." This name was used by the Greeks to designate what now is usually called mean and extreme ratio. The Greeks knew little of modern algebra and nothing of the Hindu notation. Hence their proofs are very much more difficult in appearance than the proof given of Problem 3. The actual geometrical construction, however, follows in § 346.

The numerous applications of the "golden section" in architecture and other forms of art indicate that this division of a line possesses æsthetic as well as mathematical significance.

Problem 4 *(Construction)*

346. *Divide a given line internally in extreme and mean ratio.*



Given the line ***AB.***

Required *to locate a point K on AB such that* $AB : AK = AK : KB$.

Construction. Bisect AB at L. Construct $BM \perp$ to AB at B. Lay off BG equal to BL. Construct a circle with G as the center and BG as the radius. Draw AG, cutting the circle in F and H. Lay off AK on AB equal to AF.

Then K is the required point.

Proof.	$AH : AB = AB : AF.$	(1)	§ 292
From (1),	$(AH - AB) : AB = (AB - AF) : AF.$	(2)	§ 302
Now	$AB = FH,$	(3)	Why?
and	$AF = AK.$	(4)	Why?

Substituting from (3) and (4) in (2) gives

$$(AH - FH) : AB = (AB - AK) : AK. \quad (5)$$

But	$AH - FH = AF$ or $AK,$	(6)	
and	$AB - AK = KB.$	(7)	

Therefore, using (6) and (7), (5) becomes

$$AK : AB = KB : AK, \quad (8)$$

or	$AB : AK = AK : KB.$	(9)	Why?
Therefore	K is the required point.		§ 344

347. Decagon. A *decagon* is a polygon having ten sides.

Problem 5 (*Construction*)

348. *Inscribe a regular decagon in a given circle.*

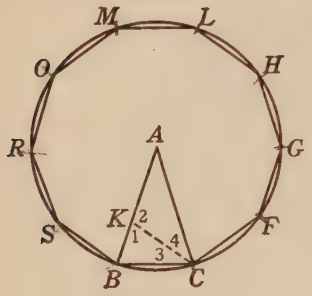


FIG. 1

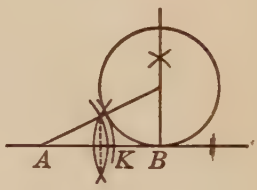


FIG. 2

Given a circle whose center is *A*.

Required *to inscribe a regular decagon in the circle.*

Construction. Draw any radius AB. Then divide AB internally in mean and extreme ratio as indicated in Fig. 2. With AK, the *larger* segment, as the radius and B of Fig. 1 as the center, describe an arc cutting the circle at C. With C as the center and AK as the radius, describe an arc cutting the circle at F. In like manner obtain points G, H, L, M, O, R, and S. Draw chords BC, CF, etc. They will form the required decagon.

Proof. In Fig. 1 draw AC and lay off AK on radius AB equal to AK of Fig. 2 and draw KC. Then in Fig. 1,

	$AB:AK = AK:KB.$	(1)	Why?
But	$BC = AK.$	(2)	Why?
Hence	$AB:BC = BC:KB.$	(3)	
Now	$\angle B = \angle B.$	(4)	
Hence	$\triangle ABC \backsim \triangle CBK.$	(5)	§ 274
Then	$AB:AC = CB:CK.$	(6)	Why?
But	$AB = AC.$	(7)	Why?

Hence	$BC = CK.$	(8)	§ 135
Then from (2) and (8),	$CK = AK.$	(9)	
Now	$\angle A + \angle B + \angle BCA = 180°.$	(10)	Why?
From (7),	$\angle B = \angle BCA.$	(11)	§ 29
From (5),	$\angle 3 = \angle A.$	(12)	Why?
From (9),	$\angle 4 = \angle A.$	(13)	Why?
Also	$\angle BCA = \angle 3 + \angle 4.$	(14)	
By (12), (13), and (14),	$\angle BCA = 2\angle A.$	(15)	
By (11) and (15),	$\angle B = 2\angle A.$	(16)	

Substituting from (15) and (16) in (10) gives

$$\angle A + 2\angle A + 2\angle A = 180°.$$

Therefore $\angle A = 36°,$

and arc BC is one tenth of the circle. Why?

Then arc $CHSB$ is nine tenths. Hence arc SB is one tenth of the circle.

Therefore $BCFGH \cdots$ is a regular inscribed polygon. Why?

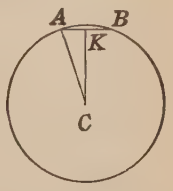
QUERY. How can a regular pentagon be inscribed in a circle? a regular polygon of twenty sides? of forty? of eighty? a five-pointed star?

EXERCISES

In the next four exercises the results of any one may be used, if possible, in any one following.

23. Show that the side of a regular inscribed decagon is $\frac{R}{2}(\sqrt{5} - 1)$.

HINT. Study equation (7) of Problem 3 and its results.

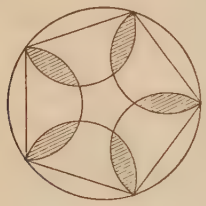


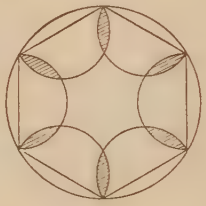
24. Show that the apothem of a regular inscribed decagon is $\frac{R}{4}\sqrt{10 + 2\sqrt{5}}$.

HINTS. In the adjacent figure what is the value of AB, the side of a regular inscribed decagon, in terms of R? of AK in terms of R? Then use the triangle ACK.

25. Show that the area of a regular inscribed decagon is $\frac{5R^2}{4}\sqrt{10 - 2\sqrt{5}}$.

26. If the radius of a circle is 8 units, show that the side of its regular inscribed decagon is approximately 4.94 units.



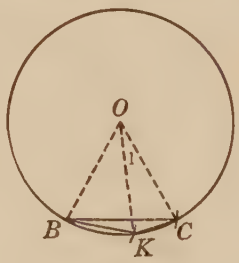


27. Construct the two designs which are given above.

QUERY. How can a regular inscribed pentagon be constructed?

Problem 6 *(Construction)*

349. *Inscribe a regular pentadecagon (15 sides) in a given circle.*



Given a circle whose center is *O*.

Required *to inscribe a regular 15-sided polygon in the circle.*

The construction and proof are left to the student.

HINTS. Let BC be the side of a regular inscribed hexagon and BK the side of a regular inscribed decagon. How many degrees are there in $\angle 1$? What fraction of the circumference is arc KC?

Exercises

28. How can a regular polygon of 30 sides be constructed? of 60 sides? of 120 sides?

29. How many regular polygons of less than 1025 sides can be constructed by the use of Problems 1–6, inclusive?

30. Write in order, the smallest first, the series of numbers involved in the answer to the preceding exercise.

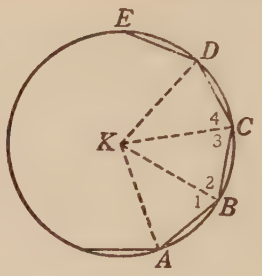
Historical Note. In the latter part of the eighteenth century Gauss (1777–1855) gave considerable attention to the subject of regular polygons. When he was but nineteen years old he devised a method of constructing a regular polygon of seventeen sides. Later he showed that if the number of sides of a polygon is a prime number and expressed by the formula $2^n + 1$, the polygon can be constructed by ruler and compasses. By bisecting the arcs subtended by the sides of each regular inscribed polygon thus obtained other regular polygons having 2, 4, 8, 16, etc. times as many sides can be constructed. Allowing for this, Gauss expressed his results by the more general formula $2^m(2^n + 1)$, where m is any positive integer or zero and n is zero or any integer which makes $2^n + 1$ a prime number. But m and n must not be zero at the same time.

Besides Gauss's construction of the regular 17-sided polygon at least three others have been devised: one by J. A. Serrett (1819–1885), one by A. L. Crelle (1780–1855), and one in 1897 by L. Gerard. The first five prime numbers given by the formula are 3, 5, 17, 257, and 65,537. F. J. Richelot carried out the solution for the regular polygon of 257 sides, and Oswald Hernes (1826–1909), after ten years of labor, completed the solution for a regular polygon of 65,537 sides. Since $2^n + 1$ in Gauss's formula must be a prime number, the number of polygons possible is a difficult problem in the theory of numbers. Very recently Professor L. E. Dickson has shown that in $2^n + 1$ if $n \leqq 100$, the number of regular polygons which may be inscribed in a circle is 24, for $n \leqq 1000$ it is 52, and for $n \leqq 100{,}000$ it is 206.

Exercise **31.** Form a table showing that Gauss's formula $2^m(2^n + 1)$ includes all the polygons which are given in the answer to Ex. 30, except those having 15, 30, etc. sides.

Theorem 1

350. *A circle can be circumscribed about any regular polygon.*



Given any regular polygon *ABCDE* · · ·.

To prove *that a circle can be described passing through the points A, B, C, D, E,* etc.

Proof. Let a circle be described through A, B, and C, point K being its center.

	Draw the radii KA, KB, KC, and the line KD.	§ 224
	$\angle 1 + \angle 2 = \angle 3 + \angle 4$.	Why?
	$\angle 2 = \angle 3$.	Why?
Therefore	$\angle 1 = \angle 4$.	Why?
	$AB = CD$.	Why?
	$KB = KC$.	Why?
Hence	$\triangle AKB$ is congruent to $\triangle CKD$,	Why?
and	$AK = KD$.	

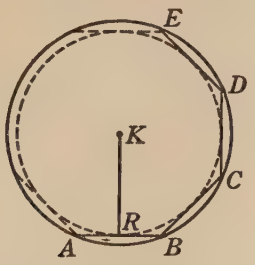
Therefore the circle K passes through D. In like manner it can be proved that the circle K passes through the other vertices.

Exercise **32.** An equiangular polygon inscribed in a circle is regular if it has an odd number of sides.

351. Radius of polygon. The radius of the circumscribed circle of a polygon is called the *radius of the polygon.*

Theorem 2

352. *A circle can be inscribed in any regular polygon.*



Given any regular polygon *ABCDE* · · ·.

To prove *that a circle may be drawn tangent to* AB, BC, CD, etc.

Proof. Circumscribe a circle about $ABCDE \cdots$. Let K be its center.

Now $AB = BC = CD = DE$, etc. Why?

These chords are the same distance from the center. Why?

Draw the apothem KR.

With K as the center and KR as the radius, a circle can be drawn which is tangent to AB, BC, CD, etc. § 191

353. ***Corollary.*** *The centers of the inscribed and circumscribed circles of a regular polygon coincide.*

354. Center of polygon. The *center of a regular polygon* is the common center of its inscribed and circumscribed circles.

355. Angle at the center of polygon. The angle at the center of a regular polygon is the angle between the two radii drawn to the extremities of any side.

Exercises

33. The angle at the center of a regular polygon is the supplement of an angle of the polygon.

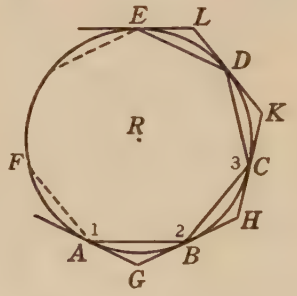
34. The apothem of an equilateral triangle is one half the radius.

Theorem 3

356. *If a circle is divided into three or more equal arcs,*

(1) *the chords joining the adjacent points of division form a regular inscribed polygon;*

(2) *tangents at the points of division form a regular circumscribed polygon.*



Given the circle R; points A, B, C, D, E, etc., dividing it into equal arcs; chords AB, BC, etc.; and tangents at A, B, C, etc., intersecting at G, H, K, L, etc.

(1) ***To prove*** *that $ABCDE\cdots$ is a regular inscribed polygon.*

Proof. Chord AB = chord BC = chord CD, etc. Why?

Hence the inscribed polygon AD is equilateral. Why?

Now $\angle 2$ is measured by one half arc AEC, or by half the whole circle minus half the sum of the two equal arcs AB and BC. Why?

Also $\angle 3$ is measured by one half of arc BFD; that is, by half the whole circle minus half the sum of two arcs each equal to arc BC or arc AB. Why?

Hence $\angle 2 = \angle 3$.

In like manner $\angle 2 = \angle 1$, etc.,

and the inscribed polygon $BCD\cdots$ is equiangular.

Therefore the inscribed polygon $BCD\cdots$ is regular. Why?

(2) ***To prove*** *that* $GHKL \cdots$ *is a regular circumscribed polygon.*

Proof. Arc $DE =$ arc DC. Why?

$\angle LED = \angle LDE$ and $\angle KDC = \angle KCD$, Why?

$\angle LDE = \angle KDC$, Why?

and chord $DE =$ chord DC. Why?

Therefore $\triangle EDL$ is congruent to $\triangle DCK$ and each triangle is isosceles. Why?

Hence $\angle L = \angle K$. Why?

In like manner $\angle K = \angle H = \angle G$, etc.,

and the polygon $GHKL \cdots$ is equiangular.

Now $EL = LD = DK = KC = CH = HB$, etc. Why?

Then $LK = KH = HG$, etc.

Therefore the polygon $GHKL \cdots$ is equilateral.

Therefore the circumscribed polygon $GHKL \cdots$ is regular. Why?

QUERY 1. At what point in the proof of Theorem 3 could § 223 have been used?

QUERY 2. Was § 193 used in the proof of Theorem 3?

QUERY 3. Can a proof that $GHKL$ is regular be based on these two sections?

EXERCISES

35. The side of the circumscribed equilateral triangle of a circle is twice that of the inscribed equilateral triangle.

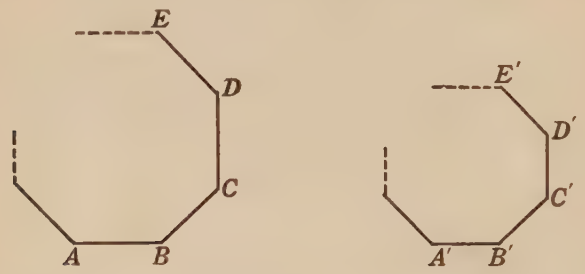
36. An angle of a regular circumscribed polygon is the supplement of the angle measured by the smaller arc intercepted by two adjacent sides.

37. An equiangular polygon circumscribed about a circle is regular.

38. An equilateral polygon circumscribed about a circle is regular if it has an odd number of sides.

Theorem 4

357. *If each of two regular polygons has n sides, the polygons are similar.*



Given the regular polygons AD and $A'D'$, each having n sides.

To prove *that AD is similar to $A'D'$.*

Proof. Since the polygons are regular,

$$AB = BC = CD, \text{ etc.,} \qquad (1)$$

and

$$A'B' = B'C' = C'D', \text{ etc.} \qquad (2)$$

From (1) and (2), $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$, etc. (3) Why?

By hypothesis each polygon is equiangular.

Hence $\angle A = \dfrac{(2n-4)\text{ rt. }\measuredangle}{n}$, (4) § 125

and $\angle A' = \dfrac{(2n-4)\text{ rt. }\measuredangle}{n}$. (5) Why?

From (4) and (5), $\angle A = \angle A'$. (6)

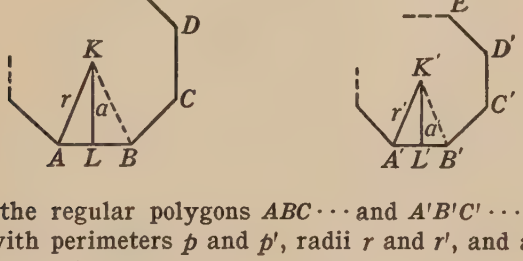
Similarly, $\angle B = \angle B'$, $\angle C = \angle C'$, etc. (7)

Therefore from (3), (6), and (7), polygons AD and $A'D'$ are similar. Why?

Exercise **39.** The diagonals of a regular pentagon form another regular pentagon.

Theorem 5

358. *The perimeters of two regular polygons of the same number of sides have the same ratio as their radii or their apothems.*



Given the regular polygons $ABC\cdots$ and $A'B'C'\cdots$, each of n sides, with perimeters p and p', radii r and r', and apothems a and a' respectively.

To prove that $$\frac{p}{p'}=\frac{r}{r'}=\frac{a}{a'}.$$

Proof. Let K and K' be the respective centers of the inscribed circles. Draw KB and $K'B'$.

$\triangle ABK \backsim \triangle A'B'K'$. Why?

Then $$\frac{r}{r'}=\frac{AB}{A'B'}.$$ (1) Why?

$\triangle AKL \backsim \triangle A'K'L'$. Why?

Then $$\frac{r}{r'}=\frac{a}{a'}.$$ (2) Why?

From (1) and (2), $$\frac{r}{r'}=\frac{a}{a'}=\frac{AB}{A'B'}.$$ (3) Why?

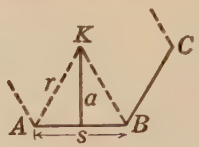
Now $$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}, \text{ etc.}$$ (4) Why?

$$\frac{AB+BC+CD+\text{etc.}}{A'B'+B'C'+C'D'+\text{etc.}}=\frac{AB}{A'B'}=\frac{p}{p'}.$$ (5) § 276

From (3) and (5), $$\frac{p}{p'}=\frac{r}{r'}=\frac{a}{a'}.$$

Theorem 6

359. *The area of a regular polygon is one half the product of its perimeter and its apothem.*



Given a regular polygon $ABC\cdots$ **of** n **sides with apothem** a **and perimeter** p.

To prove *that the area of* $ABC\cdots$ *is* $\frac{a \times p}{2}$.

Proof. Let K be the center of the inscribed circle and s one side of the polygon. Draw KA and KB.

The area of $\triangle ABK = \frac{a \times s}{2}$. Why?

The polygon can be divided into n triangles, each congruent to $\triangle ABK$.

Hence the area of the n triangles $= n\frac{a \times s}{2}$, or $\frac{a}{2} n \times s$. Why?

But $n \times s = p$, the perimeter of the polygon.

Therefore the area of the regular polygon $ABC\cdots = \frac{a \times p}{2}$.

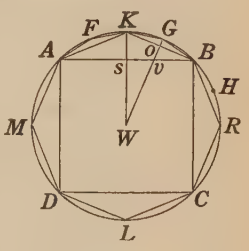
EXERCISES

40. The radius of a circle is 10″. Find the apothem, the side, and the area of its inscribed and its circumscribed squares.

41. The radius of a circle is 1. Find the apothem, the side, and the area of its regular inscribed and circumscribed hexagons.

360. Computation of the circumference of a circle. The circumference of a circle has been defined as the length of the circle. A clear notion of what is meant by the circumference and the method of computing it can be obtained as follows:

In the adjacent figure the perimeter of the square $ABCD$ is obviously less than the circumference. Now bisect the arcs AB, BC, CD, and DA at K, R, L, and M respectively, and draw chords AK, KB, BR, etc. Since $AK + KB > AB$ (Why?) it follows that the perimeter of the octagon $AKBRCLDM$ is greater than the perimeter of the square. Further, the perimeter of the octagon is less than the circumference. Now bisect the arcs AK, KB, BR, etc. at F, G, H, etc. respectively, and form the regular sixteen-sided figure $AFKGB\cdots$. Now $AF + FK > AK$. (Why?) Therefore it follows that the perimeter of the sixteen-sided polygon is greater than that of the octagon, and it also is less than the circumference of the circle. The process just outlined can be continued indefinitely and regular polygons having 32, 64, 128, etc. sides can be constructed, in each of which the perimeter is greater than the perimeter of the preceding polygons and less than the circumference.



If we assume that these polygons are constructed, it is apparent that the perimeters will increase with the number of sides of the polygons and approach as near as we please the length of the circumference of the circle.

If regular circumscribed polygons of 4, 8, 16, 32, etc. sides be constructed about the same circle, it can be shown in like manner that the perimeters of the successive polygons decrease and approach as near as we please the length of the circumference.

Further, the apothem $oW > vW >$ apothem sW. (Why?) In like manner the apothem of any of the inscribed polygons is greater than the apothem of the preceding one and in the process we are here considering can be made to approach as near as we please to the radius.

The following theorem will be assumed:

Theorem 7

361. *As the number of sides of a regular inscribed or circumscribed polygon is successively doubled, the perimeter approaches as near as we please to the length of the circumference of the circle and the apothem approaches the radius.*

362. *Corollary.* *The circumferences of two circles are in the same ratio as their radii or their diameters.*

363. Definition of π. The number π (pronounced pī), used in calculations on the circle, is the number obtained by dividing the circumference of a circle by its diameter; that is,

$$\pi = \frac{C}{D}.$$

From the above, $C = \pi D$ or $C = 2\pi R$.

364. The ratio π is the same for all circles. This follows from §§ 362 and 363.

Exercises

42. The radius of a circle is 10′. Show that the circumference is 20π feet.

43. The circumference of a circle is 32π inches. Find the radius and the diameter of the circle.

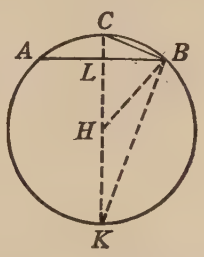
44. A circle is inscribed in a square whose side is 1′ 5″. Find the circumference in terms of π.

45. The circumference of a circle is 100π inches. Find the radius of a circle whose circumference is four times as great.

In Problem 7, which follows, a formula is derived by means of which an approximate value of π can be computed.

Problem 7 (*Computation*)

365. *If s is the side of any regular inscribed polygon, x the side of a regular inscribed polygon having twice as many sides as the first, and R the radius of the circle circumscribing both, express x in terms of s and R.*



Given the circle H; AB, or s, and BC, or x, the sides of regular inscribed polygons of n sides and $2n$ sides respectively; and the radius R.

Required *to express x in terms of s and R.*

Solution. Draw the diameter CK, cutting AB at L. Draw BK and BH.

Then CBK is a rt. $\angle$. Why?

Also CK is $\perp$ to AB. Why?

Therefore $\overline{CB}^2 = CK \times CL,$ (1) § 282

or $x^2 = 2R \times CL.$ (2)

Therefore $x = \sqrt{2R \times CL}.$ (3)

Now $CL = R - LH.$ (4)

And since $LB = \frac{s}{2}$, $LH = \sqrt{R^2 - \frac{s^2}{4}}.$ (5) Why?

From (4) and (5), $CL = R - \sqrt{R^2 - \frac{s^2}{4}}.$ (6)

From (3) and (6), $x = \sqrt{2R^2 - 2R\sqrt{R^2 - \frac{s^2}{4}}}.$ (7)

NOTE. The formula (7) is a general result useful in computing the ratio of the circumference to the diameter of a circle. Note carefully the meaning of the formula. Thus, if s is the side of an inscribed square, x is the side of a regular inscribed octagon; if s is the side of an inscribed equilateral triangle, x is the side of a regular inscribed hexagon, etc.

EXERCISES

The word *formula* in the following exercises refers to (7), § 365.

46. Read and express wholly in words:

$$s_{10} = \sqrt{2R^2 - 2R\sqrt{R^2 - \frac{(s_5)^2}{4}}};\quad s_{2n} = \sqrt{2R^2 - 2R\sqrt{R^2 - \frac{(s_n)^2}{4}}}.$$

47. In the first part of the preceding exercise if s_{20} replaces s_{10}, what other change is necessary? Make the change and express the resulting statement verbally.

48. In the second formula of Ex. 46 substitute 3 for n and express the resulting statement in words. Do the same with 6 substituted for n.

49. Use the formula to show that the side of a regular inscribed dodecagon is $R\sqrt{2-\sqrt{3}}$.

HINT. Here $s = R$ (§ 343).

50. Explain how the formula can be used to compute the side of a regular inscribed octagon in a circle of given radius.

366. Computation of π. The perimeter of a regular hexagon (or any regular polygon) inscribed in a circle is less than the circumference. The perimeter of a regular dodecagon inscribed in the same circle is greater than that of the regular inscribed hexagon, but less than the circumference; and if in the same circle as before regular polygons of 24, 48, 96, etc. sides are inscribed, the perimeter of any one is greater than the perimeter of the preceding one, but each perimeter is less than the circumference (§ 360). If we start with a circle of given radius and compute the length of one side of each

of these regular polygons and the corresponding perimeter, we shall obtain, if this process is continued, a closer and closer approximation to the length of the circle; that is, to the circumference. By dividing this perimeter by the diameter of the circle an approximate value of π can be obtained.

In the table below, n denotes the number of sides of a regular inscribed polygon, R the radius of its circumscribed circle, s the length of one side, P the perimeter, and D the diameter. $P \div D$ will be an approximate value of $C \div D$, or π.

n	R	s	P	$P \div D$ or π
6	1	1	6	$6 \div 2 = 3$
12	1			
24	1			
48	1			
96	1	.06543817	6.28206396	3.14103 +
192	1	.03272346	6.28290510	3.14145 +
384	1	.01636228	6.28311544	3.14155 +
768	1	.00818126	6.28316941	3.14158 +

R, s, and $P \div D$ for the hexagon are easily filled out, since for that polygon $s = R$ (§ 343). For the side of an inscribed dodecagon the computation is as follows: $S_6 = R = 1$. Substituting in (7) of § 365,

$$x \text{ or } s_{12} = \sqrt{2 \times 1^2 - 2 \times 1\sqrt{1^2 - \frac{1^2}{4}}} = \sqrt{2 - 2\sqrt{\frac{3}{4}}}$$

$$= \sqrt{2 - \sqrt{3}} = \sqrt{2 - 1.7320508} = \sqrt{.2679497}.$$

The most nearly accurate result as indicated in the table gives $\pi = 3.14158$. In the work as outlined above the computation of every side after the first depends on the degree of accuracy of

the computation of all those which precede. Thus, s for a polygon of 768 sides depends on s for a polygon of 384 sides, and this in turn on one of 192 sides, and so on back to the computation of the side of a dodecagon. Therefore, to insure the correctness of a final result to a certain number of decimals, the first computation must be carried farther than the later ones; that is, if the side of the polygon of 768 sides is desired correct to eight decimals, the computation of s for a dodecagon must be carried to fourteen or sixteen decimals or possibly more. Careful attention to these matters gives the value $\pi = 3.14159+$. This is usually written $\pi = 3.1416$. In work which does not require great accuracy, $\pi = 3\frac{1}{7}$ is a convenient value often used.

Exercise 51. Compute s, P, and π in the table of § 366 for the dodecagon, obtaining results to five decimals.

Historical Note. No number is more important in mathematics and its applications than the number π. The significant rôle it has played in the development of mathematics can merely be hinted at in a short note, for the history of π runs back over four thousand years. Early geometers perceived that the area of a circle is $\frac{r \times C}{2}$. They saw that a square whose side was a mean proportional between C and $\frac{r}{2}$ would have the same area as the circle, and called the problem, the solution of which required the finding of C, "squaring the circle." This is essentially the problem of determining the value of π, and it was from the considerations just stated that the attempts of mathematicians to determine π by the ruler and compasses began.

The earliest reference to π is by an Egyptian priest, Ahmes (1700 B.C.), who gives the area of a circle as $\left(\frac{8D}{9}\right)^2$, where D is the diameter. We use for the area $\frac{\pi D^2}{4}$, where $\pi = 3.1416$. Therefore $\frac{\pi D^2}{4} = \frac{64 D^2}{81}$. Solving this, $\pi = 3.1604$, the value used by the Egyptians.

The value 3 for π, probably derived from the Babylonians, is mentioned in the Bible in two places: 1 Kings vii, 23; and 2 Chronicles iv, 2.

Archimedes (born 287 B.C.), perhaps the greatest of all mathematicians except Newton, proved by a method similar to that of § 366, that the value of π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$; that is, between 3.1428 and 3.1408.

Where great accuracy is not required, $3\frac{1}{7}$ is still often used. The astronomer Ptolemy (A.D. 150) gave $\pi = 3\frac{1}{120}\frac{7}{0}$, which equals 3.14166, a value more nearly correct than that given by Archimedes.

Accurate values for π were known to the Chinese at a comparatively early date. Chang Heng (A.D. 78–139) gave $\pi = \sqrt{10}$, or 3.1623. In the fifth century A.D. Tan Chung Chin gave a value for π between 3.1415927 and 3.1415926, each value correct to the sixth decimal! By methods which he did not explain he inferred that these values were equivalent to $\frac{22}{7}$ and $\frac{3}{1}\frac{5}{1}\frac{5}{3}$ respectively; and it is a remarkable fact that $\pi = \frac{3}{1}\frac{5}{1}\frac{5}{3}$ was obtained independently eleven centuries later by Adrian Anthonisz (1527–1607), a European mathematician.

Viète (1540–1603), using polygons of $6 \cdot 2^{16}$, or 393,216, sides, found a value of π correct to nine decimals. About the same time another computer carried the work correctly to 17 decimals and another to 35 decimals.

With Newton's invention of the calculus, methods of computing the value of π were discovered far simpler than that requiring the use of inscribed polygons. By the aid of the calculus it was shown that

$$\pi = 4\,(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} \cdots).$$

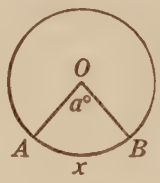
Still other series more convenient for the calculation of π were derived. Using modern methods, Shanks, in 1873, obtained the value of π to 707 decimals.

It may be observed that for a long time mathematicians endeavored to find a common fraction equal to π; this meant that they assumed that it was a rational number. Then, looking deeper, they began to suspect that π was an irrational number like $\sqrt{10}$, a value given more than once as an approximation. In 1766 Lambert gave an imperfect proof that π was irrational. In 1794 Legendre completed Lambert's proof; but the possibility of obtaining the value of π by ruler and compasses still remained, for $x = \sqrt{10} = \sqrt{2 \cdot 5}$ is easily constructed by § 305. In 1882, however, Lindeman proved that the problem of "squaring the circle," that is, of constructing with ruler and compasses a square having the same area as a given circle, was impossible.

Theoretic considerations rendered the study of the nature of π important, but the lengthy decimals with which we can now express it are never needed. For, in the words of Newcomb, "Ten decimals are sufficient to give the circumference of the earth to a fraction of an inch, and thirty decimals would give the circumference of the whole visible universe to a quantity imperceptible with the most powerful microscope."

Theorem 8

367. *The length of an arc of a circle in linear units is to the circumference as the number of degrees in the arc is to 360.*



Given the circle O whose radius is R, and the arc AB whose length is x and which contains $a°$.

To prove *that* $x : 2\pi R = a : 360.$

Proof. The length of the circle AKB is $2\pi R$. The arc x subtends a central angle of $a°$ and the circumference subtends a central angle of 360°.

Therefore $x : 2\pi R = a : 360.$ § 211

QUERY. Why are linear units specified in Theorem 8?

EXERCISES

Unless otherwise directed the value 3.1416 is to be used for π in all computations on the circle.

52. Find the length of an arc of 20° in a circle whose radius is 4.

53. An arc of a circle is 8π inches long and contains 30°. Find the radius of the circle.

54. The radius of a circle is 10 and the length of an arc is 5π. How many degrees are there in the arc?

55. A wheel stands in water which reaches halfway to the center of the hub. How many degrees are there in the arc bounding the submerged portion?

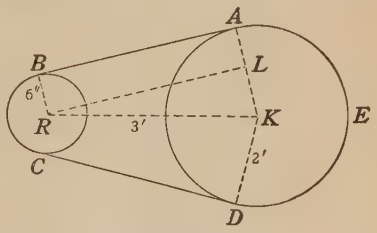
56. How long does it take the minute hand of a clock to move through 132°?

57. How long does the hour hand of a clock require to move through 87°?

58. The hands of a clock are 6′ and 7′ long respectively. How far does the outer extremity of the minute hand travel in the time from 8.30 A.M. to 1.55 P.M.? the outer extremity of the hour hand? (Use $\frac{22}{7}$ for π.)

59. The radius of a wheel on a freight car is 21 inches. If the car runs 35 miles per hour, how many revolutions does the wheel make per minute? (Use $\frac{22}{7}$ for π.)

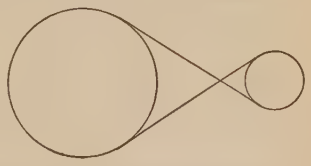
60. The centers of two pulleys (wheels) are 3′ apart. The radii are respectively 6″ and 2′. What is the length of the belt which runs around the two?



HINTS. In the adjacent figure the length of the belt is AB plus arc BC plus CD plus arc DEA. Draw KA, KR, and RB. Then draw RL parallel to AB, cutting KA at L.

61. Through how many degrees does a point on the equator revolve about the axis of the earth in 9 minutes? Through how many miles? (Use 3960 miles as the radius of the earth.)

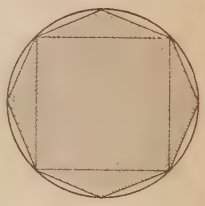
62. The radii of two pulleys are 8″ and 2″ respectively and their line of centers is 20″. Find the length of a belt passing around them and crossed as in the adjacent figure.



368. Area of a circle. The area of a parallelogram which is not a rectangle, of a triangle, or of a trapezoid depends in a simple manner on the area of a rectangle, though no one of the three figures can be completely divided into rectangles.

The area of a circle is very much more difficult to measure than that of any figure bounded by straight lines. The part of a plane bounded by a circle cannot be divided up into squares or rectangles, however small they may be, nor can its area be made to depend in any simple way on either.

Just what is meant by the area of a circle and how it may be determined can be understood by reference to the adjacent figure. The area of the inscribed square is less than the area of the circle. The area of the regular inscribed octagon is greater than that of the square but less than the area of the circle. If regular inscribed polygons of 16, 32, 64, 128, etc. sides are constructed, the area of each polygon is greater than that of the preceding one and less than that of the circle. If this process is continued, a polygon can be obtained whose area is as nearly equal to the area of the circle as we please. A similar illustration can be obtained from regular circumscribed polygons of 4, 8, 16, 32, etc. sides.



The facts, together with §§ 359–361, illustrate the truth of Theorem 9, which will be assumed.

Theorem 9

369. *The area of the sector of a circle equals one half the product of its radius and its arc.*

370. ***Corollary 1.*** *The area of a circle equals one half the product of its radius and its circumference.*

371. ***Corollary 2.*** *The area of a circle is* πR^2.

Proof. By § 363, $C = \pi D$, or $2\pi R$. Substituting $2\pi R$ for C in the formula for the area of a circle, $\frac{C \times R}{2}$ (§ 370), gives

$$\text{area of circle} = \frac{2\pi R \times R}{2} = \pi R^2.$$

372. ***Corollary 3.*** *The areas of two circles are to each other as the squares of their radii.*

QUERY 1. If a circle is divided into n equal arcs, is a sector bounded by one arc and two radii approximately a triangle?

QUERY 2. What is the altitude approximately of the triangle? the base? the area?

QUERY 3. What, then, is the approximate area of the entire circle?

QUERY 4. If n, the number of equal arcs, is doubled, are the answers to the preceding queries changed?

QUERY 5. Suppose the number of arcs is indefinitely great, what bearing have these conclusions on the truth of Theorem 9?

EXERCISES

63. If the radius of a circle is $10''$, find the circumference and the area.

64. The area of a circle is 16π. Find the diameter.

65. The circumference of a circle is 10π. Find the diameter and the area.

66. The radii of two circles are respectively 9 and 12. Find the diameter of a circle equivalent to their sum.

67. Find the radius of a circle whose circumference equals the sum of the circumferences of the two circles in Ex. 66.

68. The radii of two circles are respectively 3 and 18. Show, without finding the area of either, that the area of the larger is 36 times the area of the smaller.

69. The radius of a circle is $20''$. Find the radius of a concentric circle dividing the portion of the plane bounded by the first into two equivalent parts.

70. The area of a circle is divided by two concentric circles into three equivalent parts. The radius of the smallest circle is 10. Find the radii of the other circles.

71. Semicircles are described on the sides of a right triangle as diameters. Prove that the area of the semicircle on the hypotenuse equals the sum of the areas of the other two.

Theorem 10

373. *The area of a sector is to the area of the circle as the angle of the sector is to four right angles.*

Given the sector *AOB* in the circle whose center is *O*.

To prove *that the area of the sector* $AOB : \pi R^2 = \angle O : 360°$.

Proof. The area of the circle $= \frac{C \times R}{2}$, or πR^2. (1) § 371

The area of sector $AOB = \frac{\text{arc } AB \times R}{2}$. (2) § 369

From (1) and (2),

area of sector AOB : area of circle $= \frac{\text{arc } AB \times R}{2} : \frac{C \times R}{2}$. (3)

Or area of sector AOB : area of circle = arc $AB : C$. (4)

But arc $AB : C = \angle O : 360°$. (5) § 367

From (4) and (5), area of sector $AOB : \pi R^2 = \angle O : 360°$.

Note. There is a close analogy between the area of a sector and that of a triangle. In applying this formula care must be exercised to avoid the error of multiplying the arc in degrees by the radius in linear units.

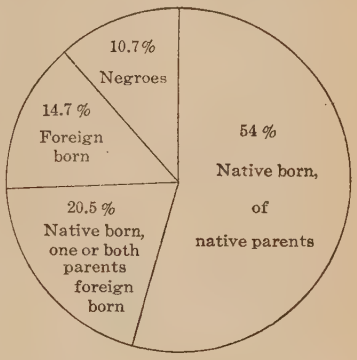
Exercises

72. The 1910 Census Report of the United States shows each of four parts which make up the entire population to be as follows: (1) native born, of native parents, 54%; native born, with one or both parents foreign born, 20.5%; foreign born, 14.7%; and negroes, 10.7%.

Display the above data by assigning to each part that sector of a circle which is its correct portion of the entire population.

Solution. Since the areas of the sectors are to each other as their central angles, we must first find the central angle which corresponds to each percentage given. Now 54% of 360° = 194°, 20.5% of 360° = 74°, 14.7% of 360° = 53°, and 10.7% of 360° = 39°.

The central angles are then constructed, as in the adjacent figure.



73. The same report gives the following percentages for the population of Dayton, Ohio: native born, of native parents, 62%; native born, with one or both parents foreign born, 21.9%; foreign born, 11.9%; and negroes, 4.2%.

Display the above data as before.

74. For Des Moines, Iowa, the census report gives the following facts: native born, of native parents, 62.3%; native born, with one or both parents foreign born, 22.3%; foreign born, 12%; and negroes, 3.4%.

Display the above data.

75. What fraction of the area of a circle is the area of a sector in it whose angle is 18°? 30°? 45°? 24°? 108°?

76. The area of a circle is 100 square feet. What is the area of a sector of 36° in this circle? of 60°? of 135°? of 40°?

77. Find the area of a sector of 20° in a circle whose radius is 10′.

78. The area of a sector is 12π square inches in a circle whose radius is 6″. Find the angle of the sector.

79. Find the circumference of a circle in which a sector of 40° has an area of 16π square feet.

80. The radius of a circle is 7′. An arc in it is 4′ long. Find the area of the corresponding sector. (Use $\frac{22}{7}$ for π.)

81. The hour hand of a clock is 8 feet long. Find the area of the sector over which it moves in the time from 12.18 P.M. to 3.40 P.M.

Theorem 11

374. *The area of a segment whose arc is less than a semicircle equals the area of the corresponding sector minus the area of the triangle bounded by two radii and the chord of the segment.*

The proof is left to the student.

QUERY. How is the area of a segment whose arc is greater than a semicircle found?

EXERCISES

82. Find the area of a segment whose arc is 90°, the radius of the circle being 10′.

83. In the preceding exercise let the arc be 30° and solve.

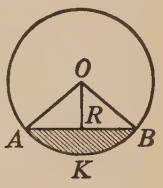
84. Let the radius of a circle be 20″ and x be the number of degrees in the arc of a segment. Find the area of the segment when x is 45°, 60°, 120°, 150°, 300°.

85. Three equal circles, radius 10″, are tangent each to the other two. Find the area of the triangular portion of the plane bounded by arcs of the three circles.

86. Show that the area of the segment bounded by a side of an inscribed square and its arc is $\frac{R^2}{4}(\pi - 2)$.

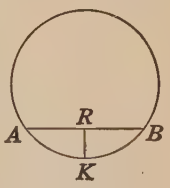
87. Show that the area of the segment of any circle bounded by a side of a regular inscribed hexagon and its subtended arc is $R^2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$.

375. Graphical determination of the area of a segment of a circle. Note that by § 373 the area of a sector can always be found if its arc or central angle and the radius of the circle are given. But given its arc or central angle and the radius of the circle, the area of a segment cannot be found by means of § 374 and the geometry so far presented unless the angle is 30°, 36°, 45°, 60°, 72°, 90°, 120°, 135°, or 150°. For example, if the radius is 10 and the arc of the segment is 80°, we must have recourse to graphical methods or to trigonometry.



To solve the problem just stated by the graphical method we can draw the circle and the segment to scale, as in the adjacent figure. Then measurement of the base AB and the altitude OR of triangle ABO will enable one to determine its area. Since the area of the sector AOB is $\frac{2}{9}$ of $100\,\pi$, the area of the segment AKB easily follows.

376. Approximate formula for the area of a segment of a circle. Frequently in practice it is much easier to measure AB, the chord, or the base of the segment in the adjacent figure, and RK, the mid-perpendicular of chord AB, than to measure the arc AKB or its central angle. Let b denote AB and h denote RK. Then the area of $AKBR$ is $\frac{h^3}{2\,b} + \frac{2\,bh}{3}$.



This formula is approximate, but gives very accurate results.

Exercises

88. Find graphically the area of a segment whose arc is 112° in a circle whose radius is 10″.

89. Find graphically the area of a segment whose arc is 280° in a circle whose radius is 20′.

90. Test the accuracy of the formula of § 376, thus: Find the area of a semicircle in the usual manner, then by the use of the formula, and compare the two results.

91. Test the accuracy of the formula of § 376 on segments whose arcs are 90° and 60° respectively.

92. Find by § 374 and by § 376 the area of the segment whose chord is the side of a regular inscribed hexagon and compare the two results.

93. The radius of a circle is 10′. The height of a segment in it is 2′. Find the area of the segment.

Miscellaneous Exercises

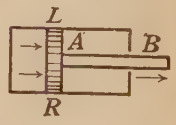
94. A central angle whose intercepted arc is equal to the radius of the circle is called a *radian*. Show that one radian equals 57.29 + degrees.

95. Show that π radians equal 180°.

96. How many degrees are there in $\frac{\pi}{2}$ radians? $\frac{\pi}{6}$? $\frac{3\pi}{2}$? $\frac{2\pi}{3}$? $\frac{\pi}{3}$?

97. How many radians are there in 90°? 135°? 60°? 240°? 30°? 45°? (Give answers in terms of π.)

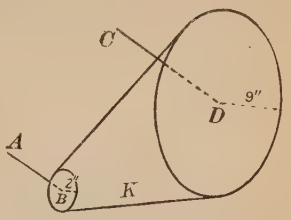
98. The diameter of a cylinder is 22″. Find the pressure on the piston LR when the steam pressure is 90 pounds to the square inch. What is the thrust along the piston rod AB?



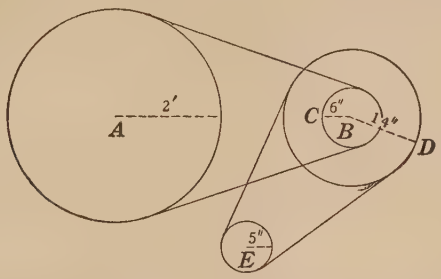
99. The earth turns on its axis once in about 24 hours. Through how many degrees does a point on the equator rotate in 5 hours? Through how many miles does it rotate about the earth's axis? (R = 4000 miles.)

100. If the earth rotates on its axis once in 23 hours and 56 minutes, how many miles does a point 30° north of the equator move in 2 hours and 20 minutes? (R = 3960 miles.)

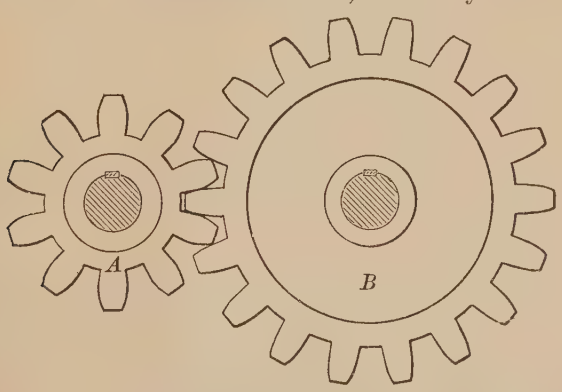
101. A continuous belt runs around two pulleys, B and D, which are mounted so as to turn freely with the shafts AB and CD, and so that if either wheel rotates the other does also. If pulley D makes 1200 R.P.M. (revolutions per minute), how many does B make?



102. If point K on the belt travels 14 feet per second, how many R.P.M. does each wheel make?

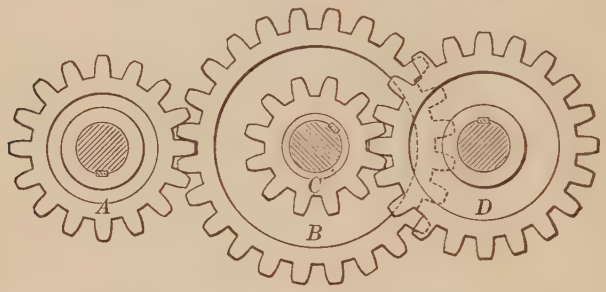


103. Belting runs around pulleys A and BC. Pulleys BC and BD are firmly fastened to the same shaft. Pulley BD is belted to E. If A makes 1400 R.P.M., how many does E make?



104. How many times does cogwheel A turn while B turns once? As A turns 30 times, how often does B turn?

105. In the figure below, the cogwheels B and C turn together, being made of a single piece of iron. While A revolves 36 times how often does the wheel B revolve? the wheel C? the wheel D?



106. While D revolves 20 times how often does C revolve? B? A?

107. While B revolves 60 times how often does A revolve? C? D?

Historical Note. In one year the student has obtained a fairly complete notion of the fundamentals of plane geometry which were known to the Greeks. During the last three hundred years, however, the labors of various mathematicians have built up the vast structure of modern geometry — an extensive body of mathematics using principles and methods of which Euclid never dreamed.

Euclidean geometry concerns itself largely with measurement; in fact, it is often called metric geometry. Only occasionally, as in Theorem 16, page 110, does it deal with a descriptive theorem — one into which no notion of measurement enters. On the other hand, modern geometry, or, more accurately, modern projective geometry, deals largely with descriptive theorems.

Regarded by some as the greatest geometrician since Euclid, Jacob Steiner (1796–1863) was perhaps the foremost contributor to the development of modern geometry. He was the son of a poor farmer and did not learn to read until he was thirteen. For a time he studied with Pestalozzi, the famous exponent of certain methods of primary teaching. Later he was a student in the University of Heidelberg. The publication of some of his original work began in 1826. Other able mathematicians

recognized his genius and succeeded in obtaining his election in 1834 to a special chair of geometry in the University of Berlin. Of other lines of mathematics Steiner remained in practical ignorance all his life. It is said that his knowledge of algebra went only as far as the quadratic equation.

Some incidents of the life of Jean Victor Poncelet (1788–1867), another eminent worker in the field of modern geometry, are both interesting and instructive. Educated in the Lyceum at Metz and in the Polytechnic School of Paris, he obtained a commission in the French engineers. During Napoleon's retreat from Moscow in 1812 he was left for dead on the field of Krasnoi. Captured by the Russians, he was imprisoned in Saratov. There, in poor quarters, with no books and with only what mathematics he remembered from his work in school, he began a very remarkable series of original researches, recording his discoveries on bits of paper brought him surreptitiously by his jailer. On his release from prison in 1814 he took with him the brilliant results of his solitary work, and in 1822 they were published under the title "Treatise on the Projective Properties of Figures," a work which will always remain a classic.

SUPPLEMENTARY EXERCISES

BOOK I

1. If the base of an isosceles triangle is divided into three equal parts, lines drawn from the vertex to the points of division are equal.

2. ABC is an equilateral triangle and K, R, and L are points on the sides AB, BC, and CA respectively such that $AK = BR = CL$. Prove that the triangle KRL is equilateral.

3. If ABC is an equilateral triangle and AB, BC, and CA are produced the same distance to K, R, and L respectively, the triangle whose vertices are K, R, and L is equilateral.

4. If each side of an equilateral triangle is divided into three equal parts and the points of division nearest to each vertex are joined respectively, a hexagon is formed which is equilateral and equiangular.

5. $ABCDEF$ is a regular hexagon. The lines AE, AD, and AC are drawn. Prove that the triangle ADE is congruent to the triangle ADC.

6. Two isosceles triangles are equal if the base and the vertex angle of one are equal respectively to the base and the vertex angle of the other.

7. ABC and ABK are two equilateral triangles on opposite sides of the same base AB. The lines BR and BL are the bisectors of the angles ABC and ABK respectively, meeting AC and AK in R and L respectively. Draw RL and prove that the triangle BRL is equilateral.

8. K is the middle point of the side AC of the equilateral triangle ABC and KR is perpendicular to BC at R. Prove that RC equals one fourth of BC.

9. If from the vertex of one of the equal angles of an isosceles triangle a perpendicular is drawn to the opposite side, it makes with the base an angle equal to one half the vertical angle of the triangle.

HINT. Draw an auxiliary line from the vertex.

10. Investigate the proof of the preceding exercise and determine whether the theorem is true when the vertical angle is acute and when it is obtuse.

11. If from any point K in one of the equal sides AB of an isosceles triangle ABC, KR drawn perpendicular to the base BC meets CA produced at R, prove that the triangle AKR is isosceles.

12. AC is the base of an isosceles triangle ABC. Any distance AL is laid off on AB. On BC produced CR is laid off equal to AL. If RL cuts AC at K, prove that KL equals KR.

HINT. Draw one auxiliary line from L parallel to AC.

13. ABK, BCL, and CDR are equilateral triangles drawn so that the sides AB, BC, and CD of the parallelogram $ABCD$ are sides of the respective triangles. The triangle BCL overlaps the parallelogram; the others are adjacent to it. Prove that RL and KL are equal respectively to the diagonals of the parallelogram. Investigate the case when two triangles overlap the parallelogram and one is adjacent, and determine if the corresponding theorem is still true.

14. The line joining the mid-points of the equal sides of an isosceles triangle is perpendicular to the median drawn to the base.

15. If two sides of a triangle and a median to one of those sides are equal respectively to two sides and the corresponding median of another triangle, the triangles are congruent.

16. Any straight line drawn from any vertex to the opposite side of a triangle is bisected by the line which joins the middle points of the other sides of the triangle.

17. In any triangle ABC the line AK is perpendicular to BC, meeting it at K. The point R is the mid-point of AK and L of AC. Prove that the triangle RKL is congruent to the triangle RAL.

18. In the preceding exercise if H is the mid-point of KC, prove that the triangle KHL is congruent to the triangle CHL.

19. If two medians are drawn from two vertices of a triangle and produced their own length beyond the opposite sides, and these extremities are joined to the third vertex, these two lines will be equal and in the same straight line.

20. In the triangle ABC a line from K on AC to R on BC makes AK equal one third of AC, and BR equal one third of BC. Prove that KR is parallel to AB.

HINTS. Draw a parallel to AB through K and one through L, the mid-point of KC, meeting BC in M and N respectively. Then prove that the points M and R are identical.

21. The point K on CB of the triangle ABC is taken so that CK equals one third of CB. Similarly, R is chosen on CA so that CR equals one third of CA. Prove that KR is one third of AB.

22. State the converse of Theorem 35, Book I. Is it true? Would this converse, if true, have been of any service in demonstrating the three preceding exercises?

23. The mid-points of two opposite sides of a quadrilateral which is not a parallelogram and the mid-points of the diagonals are the vertices of a parallelogram.

24. The lines joining the mid-points of two opposite sides of a quadrilateral which is not a parallelogram and the mid-points of the diagonals bisect each other.

25. If K and R are respectively the feet of the perpendiculars from A on the lines bisecting the angles B and C of the triangle ABC, show that KR is parallel to BC.

26. If four consecutive points, A, B, C, and D, which are in a straight line are joined to an outside point K, then (1) the angle ABK is greater than the angle ACK; (2) the angle ABK is greater than the angle CKD; (3) the angle ABK is greater than the angle ADK.

27. In the triangle ABC, if AC is equal to or less than AB, show that any straight line drawn through the vertex A and terminated by BC is less than AB.

HINT. Designate the foot of the line drawn as above by K. First compare the angle C with the angle B and then compare the angle AKB with the angle B.

28. In the triangle ABC, having the side AB greater than the side BC, the median BK is drawn. Is the angle AKB always an obtuse angle?

29. If in the triangle ABC, having the side AB greater than the side AC, the median AK is drawn, any point on AK except K is nearer to C than to B.

30. The sum of the medians of a triangle is less than the sum of the three sides of the triangle.

31. In the triangle ABC, AB is greater than BC, and BK, the bisector of the angle B, meets AC in K. Prove that AK is greater than KC.

HINT. On BA lay off BR equal to BC and draw KR. Then compare KC with KR and the angle A with the angle ARK.

32. The sum of the diagonals of any quadrilateral is less than the sum of the four sides but greater than half that sum.

33. From the mid-point K of the base AB of the isosceles triangle ABC a line is drawn to cut BC in R and AC produced in L. Prove that CR is less than CL.

BOOK II

EXERCISES ON CIRCLES

34. A parallelogram inscribed in a circle is a rectangle.

35. If an isosceles triangle is inscribed in a circle, the tangent at its vertex makes equal angles with two of its sides and is parallel to the third side.

36. The mid-perpendiculars of the sides of an inscribed quadrilateral pass through a common point. Is this true for inscribed polygons of more than four sides? Prove.

37. The base angles of an inscribed trapezoid are equal.

38. The radius of a circle inscribed in an equilateral triangle is equal to one third the altitude of the triangle.

39. What is the shortest line that can be drawn from a given external point to a circle? the longest line external to the circle? Prove.

40. The perpendicular from one vertex to the opposite side of an equilateral triangle is one and one-half times the radius of the circumscribed circle.

41. If perpendiculars are drawn to a tangent from the ends of any diameter,

(1) the point of tangency will bisect the line between the feet of the perpendiculars;

(2) the sum of the perpendiculars will equal the diameter;

(3) the center will be equally distant from the feet of the perpendiculars.

42. If an equilateral triangle is inscribed in a circle, the distance of each side from the center is equal to half the radius of the circle.

43. OA is a radius of a circle whose center is O, and B is a point on a radius perpendicular to OA. Through B the chord AC is drawn and at C a tangent is drawn meeting OB produced in D. Prove that CBD is an isosceles triangle.

44. A diameter AB is extended to K, making BK equal to the radius. KR touches the circle at R and cuts the tangent at A in L. Radius CR extended cuts the tangent AL in D. Draw KD and prove that the triangle KLD is equilateral.

45. AB is a diameter extended half its length to K. KR is a tangent, R its point of tangency. A tangent at B intersects KR in H and AR produced in L. Prove that the triangle HLR is equilateral.

46. If each of the equal angles of an inscribed isosceles triangle is double the vertical angle and its vertices the points of contact of three tangents, these tangents form an isosceles triangle each of whose equal angles is one third its vertical angle.

47. If a straight line drawn through the point of contact of two tangent circles terminates in the circles, the tangents at its extremities are parallel.

48. Two circles are tangent either externally or internally, and through the point of tangency two lines are drawn meeting one circle in B and D and the other in E and C respectively. Draw the chords BD and EC and prove them parallel.

49. If the angle between two secants intersecting outside a circle is bisected by a third secant, does the latter bisect the arcs intercepted by the first two? Prove.

HINTS. Let ABC and ALM be the secants and AKR the bisector. Let AR be on the same side of the center as AC. Draw the chords RL and RB. Now the arc BK equals the arc LK or it does not. Assume that it does. Then the arc RCB is less than the arc RML. Hence the chord RL is greater than the chord RB. Why? Fold the triangle RLA about RA as an axis upon the triangle RAB. Then L must fall on AC, but cannot fall at point B. Why? etc.

50. The line joining the mid-points of two parallel chords passes through the center of the circle.

51. Two circles intersect at A. Chords BA and KA of one circle, extended if necessary, cut the other in C and R respectively. If BK is a diameter, prove that CR is a diameter.

52. Two circles are tangent externally at A. BC is tangent to the two circles at B and C respectively. Prove that the circle described on BC as a diameter passes through A.

53. Three circles are tangent to each other at the points A, B, and C respectively. From A lines are drawn through B and C, meeting the circle which passes through B and C at the points D and E respectively. Prove that DE is a diameter.

54. If from any point on a circle perpendiculars be dropped upon the sides of an inscribed triangle (produced if necessary), the feet of the perpendiculars are in the same straight line.

Exercises in Construction

55. Construct a chord of a circle, given its mid-point.

56. Construct an isosceles right triangle, given its hypotenuse.

57. Construct an isosceles triangle, given its perimeter and its base.

58. Construct an equilateral triangle, given its altitude.

59. Through a fixed point draw a line making a given angle with a fixed line.

60. Construct a rhombus, given one side and one angle.

61. Construct a rhombus, given its perimeter and one diagonal.

62. Construct a triangle, given the angles adjacent to one side and the altitude upon it.

63. Inscribe a square in a given circle.

64. Construct an isosceles trapezoid, given the bases and the distance between them.

65. Given three fixed points K, R, and L not in the same straight line. Construct a triangle the mid-points of whose sides are K, R, and L respectively.

66. Construct a trapezoid, given two adjacent sides, the angle between them, and the difference of the bases. How many solutions are there?

67. Draw a line terminating in the equal sides of an isosceles triangle forming a quadrilateral three of whose sides are equal.

68. Construct a triangle, given two sides and the median on the third side.

HINT. Suppose ABC is the completed triangle and CK the given median. Draw KR parallel to BC to R on AC. Can the three sides of the triangle CKR be determined?

69. Construct a triangle, given the three medians.

HINTS. Let the medians AK, BL, and CR of the triangle ABC intersect at H. Draw LM parallel to AK, cutting CR at M. Can the triangle HLM be constructed?

70. Given one base and the two nonparallel sides of a trapezoid, construct the trapezoid. How many solutions are there?

71. Construct a trapezoid, given its four sides, two of which are known to be the bases. How many solutions are there?

72. From a given point on a circle draw a chord bisected by a given chord of the circle.

HINTS. Let A be the point and FG the chord. Draw two chords parallel to the chord FG so that one chord passes through A and both are equally distant from FG. Discuss the number of solutions and the limiting conditions of the problem.

73. Construct a rectangle, given the sum and the difference of two adjacent sides.

HINT. Let the sides be a and b respectively. Then $(a + b) + (a - b) = 2a$, $(a + b) - (a - b) = 2b$, etc.

74. Construct a right triangle, given the hypotenuse and the difference of the other two sides.

75. Inscribe in a given circle a triangle whose angles are equal respectively to those of a given triangle.

HINTS. Circumscribe a circle about the given triangle and draw radii to the vertices. Study the angles at the center.

76. Construct a common internal tangent to two fixed circles.

HINT. See Ex. 179, p. 150.

77. Construct a right triangle, given the hypotenuse and the sum of the other two sides.

HINTS. Let AB equal the sum. At A construct an angle KAB equal to 45°. With B as a center and the hypotenuse as a radius, describe an arc cutting AK at L. From L construct LH perpendicular to AB.

78. Construct a right triangle, given the perimeter and one acute angle.

HINTS. Let AB be the perimeter and the angle H the given angle. At A construct an angle of 45° and on the same side of AB at B an angle equal to one half the angle H. Let the sides of these angles meet in L. Construct LR perpendicular to AB. Construct LV meeting RB at V, so that the angle BLV equals one half the angle H.

79. Construct a triangle, given two of its angles and its perimeter.

EXERCISES IN LOCI

80. What is the locus of the center of a circle which touches a fixed line at a given point?

81. Find the locus of the center of a circle of given radius which touches a fixed line. Discuss.

82. Find the locus of the center of a circle which touches two fixed lines. Discuss.

83. Find the locus of the center of a circle of given radius which touches a fixed circle.

84. In a fixed circle what is the locus of the mid-points of all the chords of a given length?

85. Find the locus of the vertices of all the triangles having the common base AB and equal angles opposite AB.

86. A line cuts two fixed intersecting lines, one in K and the other in R. The four angles on both sides of KR are bisected, the bisectors meeting in H and L. If KR takes every possible position, what is the locus of H and L?

87. AB and AC, the sides of a right angle, are prolonged indefinitely. A line KR, six inches long, moves so that K remains on AB and R on AC. Find the locus of the mid-point of KR.

88. A and B are fixed points, K a variable point, on a circle. Upon a line drawn through K and A points are located on both sides of K such that their distance from K equals KB. What is the locus of these points?

HINTS. Discover the locus as was done in § 246. Then let L be a point on the locus. Draw KB and LB. Is the angle BKL constant? Is the angle L constant?

BOOK III

89. AB and AC are equal sides of a triangle. CK is the altitude on AB. Prove that $\overline{BC}^2 = 2\,AB \times BK$.

HINT. Draw the altitude on BC.

90. K is the mid-point of the arc AB. The chords KR and KL cut the chord AB in H and G respectively. Prove that $\overline{KG}^2 - \overline{KH}^2 = KH \cdot HR - KG \cdot GL$.

91. ABC is a triangle and K any point outside it. Through A' on KA produced $A'B'$ is drawn parallel to AB to meet KB produced in B'. Then $B'C'$, parallel to BC, is drawn to meet KC produced in C'. Prove that the line through C' parallel to CA will pass through A' and that the triangle ABC is similar to the triangle $A'B'C'$.

92. If any polygon replaces the triangle, will a construction like that of Ex. 91 give a polygon similar to the first?

93. K is the mid-point of BC, one of the parallel sides of the trapezoid $ABCD$. The lines AK and DK produced meet DC and AB in F and G respectively. Prove that FG is parallel to AD.

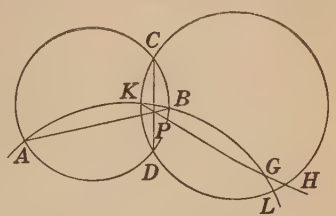
94. The two parallel lines AB and CD are joined by AD and BC, which intersect at O. On AB and CD points K and R are taken so that $AK : KB = DR : RC$. Prove that K, O, and R are in a straight line.

95. The distance of the point of intersection of the medians of a triangle from any line is one third the sum of the perpendiculars from the vertices of the triangle to the line.

96. K is any point on a circle and AB is a diameter. KR is drawn to AB and KL is drawn to BA produced, each making equal angles with AK. Prove that $AR \times BL = AL \times BR$.

97. If each of three circles intersects the other two, the three common chords are concurrent.

HINTS. Let AB and CD intersect at P. Let KP cut the circles at G and H. Then prove that G and H must coincide with L.



98. CK is the bisector of the angle C of the triangle ABC. If the angle A is equal to the angle ACK, prove that $AC : AK = AB : CB$.

99. Two circles touch internally at K. Two chords, KL and KR, of the greater circle cut the smaller circle in G and H respectively. Complete the triangles KGH and KLR and prove them similar.

100. If two circles are tangent to each other, the chords formed by a straight line through the point of contact have the same ratio as the diameters of the circles.

101. A common tangent to two unequal circles divides their line of centers, produced if necessary, into two segments which are proportional to the diameters of the circles.

102. AB is a diameter of a circle. Through A any straight line is drawn cutting the circle in C and the tangent at B in K. Prove that AC is a third proportional to AK and AB.

103. A tangent to a circle at K cuts two parallel tangents in R and L respectively. Prove that the radius of the circle is a mean proportional between KR and KL.

104. Tangents from A to a circle whose center is K touch at B and C respectively. BR is drawn perpendicular to CK produced. Prove that $AC \times BR = CR \times CK$.

105. AB and AC are any chords of a circle and AD is a diameter. The tangent to the circle at D intersects AB and AC produced at E and F respectively. Draw BC and show that the triangles ABC and AEF are similar.

106. K, R, L, and M are points on AB, BC, CD, and DA respectively of rectangle $ABCD$ such that $KR = LM$. Prove that $\overline{AL}^2 + \overline{CM}^2 = \overline{AR}^2 + \overline{CK}^2$.

107. KR is drawn from the mid-point of AC perpendicular to the hypotenuse AB of the right triangle ABC. Prove that $\overline{BR}^2 = \overline{AR}^2 + \overline{BC}^2$.

HINT. Draw BK.

108. In a right triangle h is the hypotenuse and a and b are the other two sides. The corresponding sides of another right triangle are H, A, and B. If $h : H = a : A$, prove the two triangles similar.

109. Every line parallel to the bases of a trapezoid and terminating in the nonparallel sides, but not passing through the intersection of the diagonals, is divided by the diagonals into three segments, two of which are equal.

110. In the triangle ABC a line cuts AB produced in L, BC in K, and AC in R. Prove that $AL \times BK \times CR = AR \times BL \times CK$.

HINTS. Draw perpendiculars from A, B, and C to RL, meeting it in H, M, and G respectively. Then study the triangles thus formed.

111. The length of the perpendicular to a chord, produced if necessary, from any point on the circle is a mean proportional between the perpendiculars from the same point upon the tangents drawn at the extremities of the chord.

112. From A four lines are drawn so that the angle between each adjacent pair is 45°. A straight line cuts these four in K, R, L, and H respectively, making the triangle ARL isosceles. Prove that $KR^2 = KH \times RL$.

113. A common external tangent drawn to two circles cuts the line of centers produced. Prove that radii to the points of contact form a pair of similar triangles.

114. $ABCD$ is a trapezoid, AB and CD the bases. AC and BD meet in K. Prove that $KA : KB = KC : KD$.

115. R is the mid-point of the median AK of the triangle ABC. BR extended cuts CA in L. Prove that AL is one third of AC.

116. The product of any two sides of a triangle is equal to the square of the bisector of their included angle plus the product of the segments of the third side made by that bisector.

HINTS. Circumscribe a circle about $\triangle ABC$. Extend the bisector CK to meet the circle in R and draw BR.

$$\triangle ACK \backsim \triangle CBR.$$

Therefore	$CK : CB = AC : CR.$	(1)
From (1),	$AC \times CB = CK^2 + CK \times KR.$	(2)
But	$CK \times KR = AK \times KB.$	(3)
From (2) and (3),	$AC \times CB = CK^2 + AK \times KB.$	

NUMERICAL EXERCISES

117. Two sides of a triangle are 10 and 12 respectively. The bisector of the angle between them divides the third side into segments 5 and 6 respectively. Find the length of the bisector.

118. The sides of a triangle are 20, 39, and 45 respectively. Find the length of the bisector of the angle opposite the side 39.

119. In the triangle ABC, b is 24, a is 36, and c is 30. Find to one decimal the length of the bisector of B.

120. Two sides of a parallelogram are 16 and 30 respectively and one angle is 120°. Find both diagonals.

121. In the figure of Ex. 120, if one angle of the parallelogram is 45°, find both diagonals.

EXERCISES IN CONSTRUCTION

122. In a given triangle inscribe a parallelogram similar to a given one, one of whose angles equals one of the angles of the triangle. (The vertices of the parallelogram must be on the sides of the triangle.)

HINTS. Let ABC be the triangle. Construct $AKRL$ similar to the given parallelogram, K being on AC and L on AB. Let AR cut BC in G. Then G and A are two vertices of the required parallelogram.

123. Inscribe a rhombus in a given triangle.

124. Inscribe a square in a given triangle.

125. From K on the arc AB draw KL to L on the chord AB so that KL is a mean proportional between AL and LB.

126. From a given point draw a secant terminating in a given circle so that the external segment is one half the whole secant.

HINTS. $\overline{AB}^2 = 2x^2$ etc.

127. Construct a circle touching two fixed lines and passing through a given point.

HINT. Let KA and KB be the given lines and P the given point. Draw KH bisecting the angle AKB. Draw $PM \perp$ to KH, meeting it at M. Extend PM to D, making $MD = PM$. Let MP cut AK in R. Then, if T is the required point of tangency for KA, $\overline{RT}^2 = RP \cdot RD$, etc.

128. Construct a circle passing through two fixed points and touching a fixed line.

BOOK IV

129. The area of a rhombus is one half the product of its diagonals.

130. The diagonals of a parallelogram divide it into four equivalent triangles.

131. Through the vertices of a quadrilateral straight lines are drawn parallel to the diagonals. Prove that the area of the parallelogram which they form is twice that of the quadrilateral.

132. R and K are the mid-points of CD and AB respectively of the parallelogram $ABCD$. L is any point in KR. Draw AL and DL and prove that the area of the triangle ADL is one fourth the area of $ABCD$.

133. Through K and R, the mid-points of two sides of a triangle, any two parallels are drawn cutting the third side, produced if necessary, in H and L respectively. Draw KR and prove that the area of $KRLH$ is one half the area of the triangle.

134. ABC and ABK are two equivalent triangles on opposite sides of AB. Prove that CK is bisected by the common base, produced if necessary.

135. AB and CD are the two nonparallel sides of a trapezoid and K is the mid-point of CD. Draw AK and BK and prove that the area of the triangle ABK is one half that of $ABCD$.

136. In the triangle ABC the medians AK and BR produced intersect a parallel to AB through C in L and M respectively. Draw AM and BL and prove the triangles BCL and ACM equivalent.

137. AKB and ADL are two straight lines such that the triangles AKL and ABD are equivalent. The parallelogram $ABCD$ is completed and BL is drawn cutting CD in R. Prove AK equal to CR.

HINT. KD is parallel to BL. Therefore the triangle AKD is congruent to the triangle BCR.

138. The angle A and the side AB of the triangle ABC equal respectively the angle K and the side KL of the triangle KRL. Prove that the areas of the two triangles are to each other as AC is to KR.

139. $ABCD$ is a parallelogram. Line HKL cuts AD at H and BC at L and bisects AC at K. A parallel to AD through K cuts AB in R. Draw CR, RL, and HR and prove that the triangle CRL is equivalent to the triangle AHR.

140. K is the mid-point of the diagonal BD of the quadrilateral $ABCD$. Draw AK and CK and prove that the area of the quadrilateral $ABCK$ is one half that of $ABCD$.

141. $ABCD$ is a quadrilateral. AK is drawn to K on BC. Line KR parallel to AB cuts AD (not AD produced) in R. Draw line BR and prove the quadrilateral $AKCD$ equivalent to the quadrilateral $BRDC$.

142. $ABCD$ is a quadrilateral such that K, the mid-point of the diagonal BD, lies on the same side of the diagonal AC as the vertex D. Through K a parallel to AC cuts DC in R. Draw AR and prove that it divides $ABCD$ into equivalent parts.

143. Through K, any point on a diagonal of a parallelogram, two lines are drawn parallel respectively to two adjacent sides. Prove that two of the parallelograms having a vertex at K are equivalent.

144. BK and CR are the medians from the vertices of the acute angles of the right triangle ABC. Prove that $4\,BK^2 + 4\,CR^2 = 5\,BC^2$.

Numerical Exercises

145. Find the diagonals of a rhombus whose side is 10 feet and whose area is 96 square feet.

146. The area of a rhombus is 336 and one side is 25. Find the diagonals.

147. The base of an isosceles triangle is 10, and each of its base angles is 48°. Find the altitude on the base 10.

Hint. $\dfrac{\text{Altitude}}{\frac{1}{2}\text{ base}} = \tan 48°$, etc.

148. Two sides of a triangle are 7 and 12 respectively, and their included angle is 56°. Find (1) the altitude on the side 12, (2) the area of the triangle.

149. The sides of a certain triangle are 9, 11, and 15 respectively. Find (1) the area of the triangle, (2) the altitude on the side 15, and (3) the two angles adjacent to the side 15 (using the sine formula).

150. Two adjacent sides of a parallelogram are 7 and 12 respectively. The angle between these two sides is 36°. Find the altitude on the side 12 and the area of the parallelogram.

HINT. Use the definition of the sine formula to find the altitude of the parallelogram.

151. Two adjacent sides of a parallelogram are 8 and 11 respectively. If the altitude on the base 11 is 5, find the number of degrees in each angle of the parallelogram.

152. Each of the nonparallel sides of a certain trapezoid is 8 inches. If the parallel bases are 17 and 21 respectively, find each angle of the trapezoid.

153. The diagonal of a certain rectangle is 15 inches long and makes an angle of 18° with one base. Find the dimensions of the rectangle.

154. The dimensions of a certain rectangle are 7 by 12. Find the angles which the diagonal makes with the sides of the rectangle.

CONSTRUCTION EXERCISES

155. Construct a square equivalent to a given pentagon.

156. Construct an equilateral triangle equivalent to a given square.

157. Divide a quadrilateral into two equivalent parts by a line through one vertex.

158. Divide a quadrilateral into two equivalent parts by a line through a given point on one side.

159. Divide a trapezoid into two equivalent parts by a line through any point in either base.

160. Divide a trapezoid into two equivalent parts by a line parallel to the bases.

161. Divide a parallelogram into two equivalent parts by a line through a given point in one of the bases.

162. Divide a trapezoid into two equivalent parts by a line perpendicular to the bases.

163. Construct on AB as a base an isosceles triangle equivalent to the triangle ABC.

164. Divide a triangle into two equivalent parts by a line parallel to one side.

165. Construct a triangle in which the altitude and the base are equal and which is equivalent to a given triangle.

166. Through a given point in one side of a triangle draw a line dividing the triangle into two equivalent parts.

HINT. Apply § 325 or draw the median to the side on which the given point lies.

BOOK V

NUMERICAL EXERCISES

167. In Query 17 on page 146 of Book II compute the area over which the horse can graze.

168. The radius of a circle is 10″. Show that the difference between its area and that of the regular circumscribed hexagon is 32.25 square inches.

169. The radius of a circle is 13′ and the chord of a segment is 10′. Find the area of the larger of the two segments.

170. The side of a square is 10. A circle has the same perimeter as the square. Compare the areas of the square and the circle.

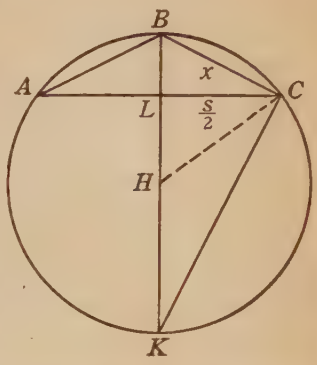
171. The hands of a clock are 10″ and 14″ respectively. How far does the outer extremity of the minute hand move in the time from 12.40 P.M. to 7.20 P.M. (Use $\frac{22}{7}$ for π.)

172. Under the same pressure how many times as much water per minute will be delivered by a pipe whose inside diameter is 6″ as by one whose inside diameter is 2″?

173. A circular tank 8 feet in diameter contains 250 tons of water. Find the pressure on one square inch of the bottom.

174. The rim of a circular saw 4 feet in diameter runs 10,000 feet per minute. How many revolutions per second does the saw make?

175. Show that the length of the side of a regular inscribed pentagon is $\frac{R}{2}\sqrt{10-2\sqrt{5}}$.



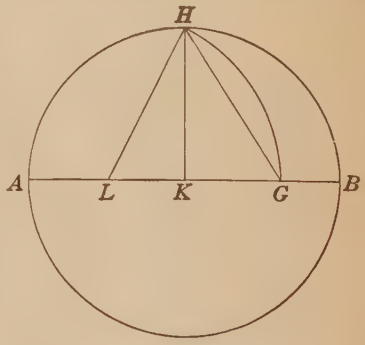
HINTS. In the adjacent figure let AB and BC be sides of a regular inscribed decagon. Then AC is the side of a regular inscribed pentagon. Let BK be a diameter and H the center. From previous work BC is known in terms of R, and BK is $2R$. Then in the triangle BCK find BL. Having BL and BC find LC, etc.

176. Show that the apothem of a regular inscribed pentagon is $\frac{R}{4}(1+\sqrt{5})$.

177. Show that the area of a regular inscribed pentagon is $\frac{5R^2}{8}\sqrt{10+2\sqrt{5}}$.

178. The radius of a circle is 10. Show that the area of its regular inscribed pentagon is 237.76 +.

179. In the adjacent figure AB is a diameter and KH a radius perpendicular to AB. L is the midpoint of AK. LG equals LH. Prove that KG is equal to the side of a regular inscribed decagon.



HINTS. Compute in terms of R lines LK, LH, LG, and KG in the order named. The last result should agree with that obtained in Ex. 23. Show similarly that HG is equal to the side of a regular inscribed pentagon.

180. From a study of the preceding exercise state a method different from that of Problem 5 for inscribing a regular decagon and a regular pentagon in a circle.

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